Lesson 01

INTRODUCTION TO MATHEMATICAL ECONOMICS

TOPIC 001: DEMYSTIFYING MATHEMATICS AND MATHEMATICAL ECONOMICS

Who's Afraid of Mathematics (The oldest friend)?

- Unity of Allah and His Book in '*Kalma*' teaches us Math.
- Inception of Life on Earth teaches us Math.
- Adam gave us 1st number.
- Eve gave us addition.
- Their children gave us multiplication.
- Total number of Prophets give us idea of large numbers.

Origin of Mathematics

From Latin '*mathēmatica'* that stands for mathematics and '*mathēmaticus'* stands for mathematical.

Age of Mathematics

"Mathematics is as old as man." (Stefan Banach)

Universality of Mathematics

Film is one of the three universal languages, the other two: mathematics and music. (Frank Capra)

The study of mathematics, like the Nile, begins in minuteness but ends in magnificence. (Charles Caleb Colton)

"Pure mathematics is, in its way, the poetry of logical ideas." (Albert Einstein)

Mathematics is the music of reason. (James Joseph Sylvester)

God used beautiful mathematics in creating the world. (Paul Dirac)

All science requires mathematics. The knowledge of mathematical things is almost innate in us. This is the easiest of sciences, a fact which is obvious in that no one's brain rejects it; for laymen and people who are utterly illiterate know how to count and reckon. (Roger Bacon)

"Without mathematics, there's nothing you can do. Everything around you is mathematics. 'Everything' around you is numbers." (Shakuntala Devi)

Human beings have a certain number of eyes, ears, hands, arms, legs, feet. And there is also a certain number of skies i.e. seven skies.

Precision via Mathematics

My belief is that nothing that can be expressed by mathematics cannot be expressed by careful use of literary words. (Paul Samuelson)

Mathematics allows for no hypocrisy and no vagueness. (Stendhal)

I've always enjoyed mathematics. It is the most precise and concise way of expressing any idea. (N. R. Narayana Murthy)

"A problem well stated is a problem half solved." (Charles Franklin Kettering)

Applicability of Mathematics

The true mathematician is not a juggler of numbers, but of concepts. (I. Stewart (1975))

Mathematics as a Subject

Despite, its primitive origin, students are usually afraid of mathematics.

Opinion of *Shakuntala Devi* (a.k.a. Human Computer), "Why do children dread mathematics? Because of the wrong approach, because it is looked at as a subject." (*Shakuntala Devi)*

Good news for young people as they like adventures. "Mathematics is an adventurous and satisfying route to doing things." Every mathematical problem starts with uncertainty of not having a definite solution but the solution, if obtained, gives a sense of fulfillment and the journey from the problem to the solution becomes an adventure.

A central figure in journey of scientific revolution is Galileo Galilei (a.k.a mathematical Platonist). He says, "If I were again beginning my studies, I would follow the advice of Plato and start with mathematics."

"Don't worry about your difficulties in mathematics. I can assure you mine are still greater". (Albert Einstein)

I've always enjoyed mathematics. It is the most precise and concise way of expressing any idea. (N. R. Narayana Murthy)

I've always been interested in using mathematics to make the world work better. (Alvin E. Roth)

This sums the learning and inspiration from the elders.

Mathematics for Economics vs. Mathematical Economics

No carving in the stone. However;

Former is about '<u>Comprehension</u>' of the mathematical tools applied to economic situations. [Algebra, matrices, differential calculus etc.]

Whereas, latter deals with the '<u>Application</u>' of mathematical tools to comprehend and solve of economic situations [e.g. application of algebra on market equilibrium, application of matrices on national income analysis, application of differentiation for marginal utility/cost/product etc].

Mathematical Economics: An Approach to Economics rather than its Branch

Can include problems from microeconomics, macroeconomics, public finance, international trade and other branches of economics.

e.g. Equilibrium for demand and supply curves, calculation of taxation revenue, local price elasticity of foreign demand of exports, among others.

<u>TOPIC 002: MATHEMATICS VS. NON-MATHEMATICAL ECONOMICS</u>

Non-mathematical economics can also be termed as literary economics.

Both remain approaches and should not fundamentally differ from each other.

However, there are two noteworthy differences:

- Former uses symbols/equations to denote assumptions and conclusions instead of words/sentences.

An Assumption in Mathematical Economics

Rather stating an assumption that a consumer can consume whole of any increase in his income, or a part of it or none of it is mentioned in mathematical fashion as follows:

$$\mathbf{0 \leq MPC \leq 1}$$

MPC = Marginal Propensity to Consume.

A Conclusion in Mathematical Economics
Rather stating a conclusion that an increase in private investment has brought about 3 times of increase in national income can be expressed in mathematical expression as follows:

$$K_I = \frac{\Delta Y}{\Delta I} = \frac{PKR30M}{PKR10M}$$

K_I= Investment multiplier

ΔY= Change (increase) in national income.
ΔI = Change (increase) in private investment.

Theorems in Mathematical Economics
Theorem: A rule in mathematics expressed in terms of symbols and formula.
Envelope Theorem: Applied to producer theory and auction theory.
Roy's Identity: Applied to Marshallian demand function.
Shephard Lemma: Used to explain relationship between expenditure (or cost) functions and Hicksian demand functions.

Advantage(s) of Mathematical Economics
- Mathematical economics utilizes the symbols that are more convenient to use in deductive reasoning.
- The use of symbols also increases conciseness and preciseness of statement.
- Use of mathematical functions allow to incorporate more than two economic variables in a situation without resorting to 3-dimensional or hyperspace graphs i.e. allows for 'n' number of variables.

Practical Importance of Mathematical Economics
- During current times, economics is said to be highly mathematized.
- Enables the comprehension of the professional articles one comes across in such periodicals as the American Economic Review, Quarterly Journal of Economics, Journal of Political Economy, Review of Economics and Statistics, and Economic Journal.

<u>TOPIC 003: MATHEMATICAL ECONOMICS VERSUS ECONOMETRICS</u>
Mathematical economics is often confused with econometrics.
- **Etymology of Econometrics:** A portmanteau of two words 'Econo' from economics and 'metrics' which implies measurement.
- **In Jargonized language:** Econometrics is the study of empirical observations using statistical methods of estimation and hypothesis testing.
- Mathematical economics is, however, concerned with application of mathematical tools on economic theory without much concern to measurement of variables and errors in it.
- Thin line between a mathematical model and econometrics model:
- $Y=\alpha+\beta X$ is a mathematical model while adding error term (ϵ) in it makes it look like:
- $Y=\alpha+\beta X+\epsilon$ which is regarded as econometric model.

Mathematical Economics	Econometrics
Application of Mathematical tools to Economic Theories	Statistical Analysis of Empirical Data of Economic Variables
Commonly used tools in Analysis: Algebra, Matrices, Differential Calculus, Integral Calculus, among others.	Commonly used tools in Analysis: Regression, Correlation, among others.
No error terms	Error terms are prerequisite
Provides theoretical Framework for Empirical Analysis	Utilizes the theoretical Framework and subjects the empirical data to analysis.
Calculate (حساب لگانا)	Estimate (تخمینہ لگانا)
Determine (احاطہ کرنا)	Infer (قیاس آرائ کرنا)

Difference with an Example
Consider law of demand and its mathematical and econometric models:

Mathematical Model
$Q_d = \alpha + \beta . P$ is a mathematical model.
Regardless of fact that many variables play a significant role in determining quantity demanded (Q_d).
These untapped variables are silenced using '*ceteris paribus' assumption.* A caveat is that other factors that can affect quantity demanded like income, price of related goods, tastes, weather, location etc. are not included in equation.

Econometric Model
While adding error term (ϵ) in it makes it look like:
$Q_d = \alpha + \beta . P + \epsilon$ which is regarded as econometric model.

- Here the assumption of '*ceteris paribus'* is relaxed which gives rise to error term (ϵ).
- It makes a grab-bag of all other factors like income, price of related goods, tastes, weather, location etc. and includes all of them in the equation as a residual.

Lesson 02

INGREDIENTS OF A MATHEMATICAL MODEL

TOPIC 004: INGREDIENTS OF A MATHEMATICAL MODEL: VARIABLES AND ECONOMIC VARIABLES

Mathematical economics resorts to models for economic analysis.

- Their building blocks are variables, constants/ coefficients and parameters.
- These building blocks are usually combined using algebra, trigonometry and other branches of mathematics to make a mathematical model.
- For example, law of demand uses algebra to connect building blocks:

$$Q_d = \alpha + \beta P$$

- Solution of Price time path in dynamic analysis uses trigonometric ratios:

$$P(t) = e^t(3\cos 2t + 2\sin 2t) + 9$$

Variables

- Etymology: (Vary + Able = Variable)
- A phenomenon that has ability to vary (usually over time).
- Life is full of specific variables: age, knowledge, weight, temperature, etc.
- However, mathematical variables are general in nature conventionally denoted by x, y, z or X, Y, Z.

Economic Variables

Some instances of economic variables:

- Demand (D)
- Supply (S)
- Price (P)
- Individual Income (Y)
- Consumption (C)
- Investment (I)
- Savings (S)
- Revenue (R)
- Costs (C)
- Profit (π)
- Wage (W)
- Gross Domestic Product (GDP)
- Government Expenditure (G)
- Taxes (T)
- Supply of Money (M_s)
- Demand for Money (M_d)
- Interest Rate (i)
- Inflation ($\dot{P}$)
- Exports (X)
- Imports (M)

Freezing a Variable

Allotting a certain value to a variable is freezing a variable.

Microeconomic Examples

- Excess supply in real market

$Q_d = 250$, $Q_s = 350$, P = 50.

Excess Supply (ES) = $Q_s - Q_d$

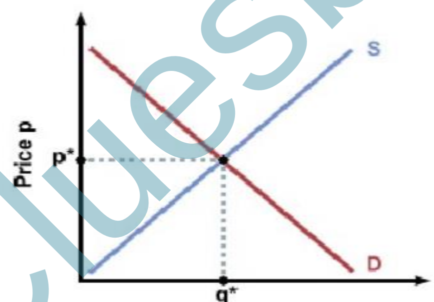
Excess Supply (ES) = 350 - 250
Excess Supply (ES) = 100 units

Profit of a Firm
R = 5,00,000, C = 3,00,000, and $\pi = R - C$, then
π = 5,00,000 - 3,00,000
π = PKR 2,00,000.
Freezing a variable is useful in calculations of other economic variables.

Macroeconomic Example
If C = 20m, I = 5m, G = 0.2m, X = 0.1m, M = 0.5m, then national income in 4-sector economy is formulated as:
GDP = C + I + G + (X-M)
GDP = 20m + 5m + 0.2m + (0.1m - 0.5m)
Gross Domestic Product (GDP) = PKR 24.8m

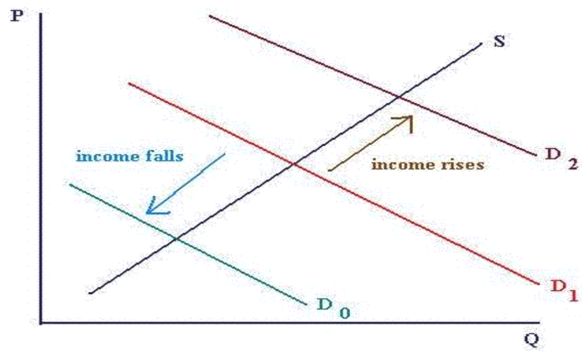
Solution Values
Properly constructed economic models give ‘solution values’.
In superjacent examples. i.e.:
- Excess Supply (ES) = 100 units
- Profit (π) = PKR 2,00,000.
- Gross Domestic Product (GDP) = PKR 24.8m

In addition, equilibrium D=S in goods market is a classic example of an economic model that gives solution values (q^*, p^*).



Endogenous vs. Exogenous Variables
Endogenous Variables
- **Etymology:** {‘Endo’ (internal) + ‘genous’ (generating)} = ‘Internally generated’.
- Variables, solution values of which comes from the model.
- e.g. Market equilibrium model uses Q_d, Q_s and P to give the values of P^* and of Q^*.

Exogenous Variables.
- **Etymology:** {‘Exo’ (external) + ‘genous’ (generating)} = ‘Externally generated’.
- Variables, values of which come from outside the model.
- Their values are assumed to be given (not influenced by endogenous variables).
- Exogenous variables shift the relationship curves of endogenous variables.
- e.g. Income can shift the demand curve either inwards or outwards.



- At macroeconomic level, political-will and policy variables like supply of money (M_s), tax rates (t) and government expenditure (G) are usually, but not always treated as exogenous variables.

TOPIC 005: INGREDIENTS OF A MATHEMATICAL MODEL: CONSTANTS AND PARAMETERS

Constants

- Etymology: Latin Origin (*Con* (with) + *Stare* (stand) = Constant)
- Lexically speaking; to stand or hold out against; resist or oppose, especially successfully.
- It's an antithesis of a Variable.
- Daily life examples: total number of, God(s) = 1 and Prophets = 1,24,000.

Constants in Mathematical/Economic Analysis

- In economic analysis, constants can appear in either numerical or symbolic manner.
- Range of numerical constants: $(+\infty\ to\ -\infty)$. e.g. 0.7, 7, -7000 etc.
- In symbolic manner, usually denoted by $a,\ b,\ c$ or $\alpha,\ \beta,\ \gamma$ or $A,\ B,\ C$.
- **Caveat:** A constant in a product with a variable, is called 'coefficient'. e.g In $7x,\ -10y$ and $ax,\ -by$. $7,\ -10,\ a,$ and $-b$ are coefficients.
- Coefficients tend to amplify or compress the efficiency of variable - hence suitably called 'co-efficient'.
- $7x$ is 7 times larger than x and $-10y$ is 10 times smaller than y.

Parameters

- **Lexical Meaning:** احاطہ / or a boundary of some activity.
- Value of parameter can change but within a restriction. Therefore, it is also called 'parametric constant'.
- Such restrictions can be called as 'parametric restrictions'.
- e.g. value of MPC can vary but from 0 to 1.
- Mathematically, it is written as:
- $(0 \le MPC \le 1)$
- Similarly, for Marginal Propensity to Save (MPS)
- $(0 \le MPS \le 1)$
- Summarily, we can say that 'parameter is constant that is somewhat variable'.

Exogenous Variables as Parameters

- For instance, in Keynes psychological law of consumption

$$\boldsymbol{C = C_o + MPC \cdot (Y)}$$

C and Y are endogenous
C_o is exogenous
MPC is parameter.

- Some writers consider exogenous variables as parameters.
- However, no carving in stone, rather a matter of convention.

TOPIC 006: A FEW ASPECTS OF LOGIC: PROPOSITIONS, IMPLICATIONS AND NECESSARY AND SUFFICIENT CONDITIONS

Logic

- Etymology: Late Latin origin (*Logica* = Art of reason)
- In mathematical reasoning Logic is to be developed.
- Two major ingredients of logical reasoning: propositions and implications.

Propositions

- Etymology {Latin: *proponere* (propound/propose)}
- Assertions that are either true or false.
- True Proposition: “All individuals who breathe are alive”.
- False Proposition: “all individuals who breathe are healthy”.

Propositions

- Imprecise proposition hinders in developing logic.
- “67 is a large number”
- Need for definition of “large number”.
- Mathematics gives precise results using algebra, matrices and calculus etc.
- Therefore, mathematical economics helps to develop clear propositions and hence better logic.

Implication(s)

- Concatenates the propositions in logical reasoning using ‘implication arrow’ (⇒).
- Let A and B be propositions and whenever *P* is true, *Q* is necessarily true.
- ‘Implication arrow’ concisely expresses:
- P ⇒ Q
- “*P* implies *Q*”, or “if *P*, then *Q*”, or “*Q* is a consequence of *P*”.

Implication(s)

More examples:

- $x > 2 \Rightarrow x^2 > 4$
- $xy = 0 \Rightarrow x = 0 \; or \; y = 0$
- x is a square ⇒ x is a rectangle

Logical Equivalence

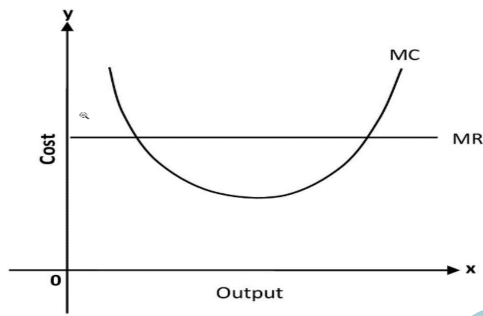
- From last slide:
- $\boldsymbol{xy = 0 \Rightarrow x = 0 \; or \; y = 0}$
- Reverse is also true, as:
- $\boldsymbol{x = 0 \; or \; y = 0 \Rightarrow xy = 0}$
- Such is logical equivalence and uses equivalence arrow (⇔):
- $\boldsymbol{xy = 0 \Leftrightarrow x = 0 \; or \; y = 0}$
- Similarly, for logical equivalence between propositions **A** and **B:**
- $\boldsymbol{A \Leftrightarrow B}$

- Read as A is equivalent to B.

Necessary and Sufficient Conditions

- Literary economics extensively uses necessary and sufficient conditions.
- In mathematical economics, we use logical equivalence to show these conditions in a concise way.
- Logical equivalence $\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$ means that $\boldsymbol{A}$ is necessary and sufficient condition for $\boldsymbol{B}$**.**

Necessary and Sufficient Conditions

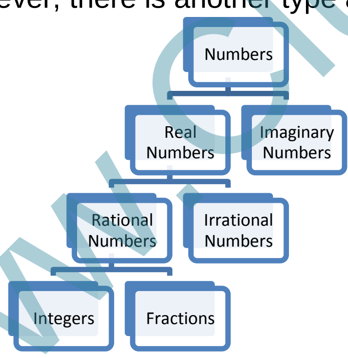


- If MC cuts MR from below, then $\boldsymbol{\pi}$ is maximum.
- $(\boldsymbol{MC\ cuts\ MR\ from\ below}) \Leftrightarrow \boldsymbol{\pi_{max}}$
- **MC cuts MR** means MC = MR
- **From below** means $Slope_{MC} > Slope_{MR}$
- Hence logical equivalence explains concisely.

<u>TOPIC 007: THE REAL-NUMBER SYSTEM</u>

Number System

- Economic variables adopt numerical values.
- Number system guides about types of numerical values.
- In this course, we deal with real numbers only.
- However, there is another type as well as shown in figure.



Real vs. Imaginary Numbers

Mostly, real number are used in economic analysis.

A Digression

- Differ from imaginary numbers: which include square root of (-1).
- This value is denoted by a Greek Letter *iota* (ι).
- $\iota = \sqrt{-1}$, so $\iota^2 = -1$
- In usual economic situations such values don't occur.

- However, in some economic situations, we have to deal with imaginary numbers. (Which is beyond the scope of this course).

Rational vs. Irrational Numbers

Rational Numbers

- Can be expressed in a ratio; $(^p/_q)$ where, $q \neq 0$.

Where, p and q are integers $(\dots, -3, -2, -1, 0, 1, 2, 3, \dots)$

- $q = 0 \Rightarrow (^p/_0) = \infty$ (undefined).
- Infinity is undefined and is hard to interpret just like timelessness.

Examples

$(^6/_2 = 3)$

$(^5/_2 = 2.5)$

$^2/_3 = 0.66666 \dots)$ same pattern repeats infinitely.

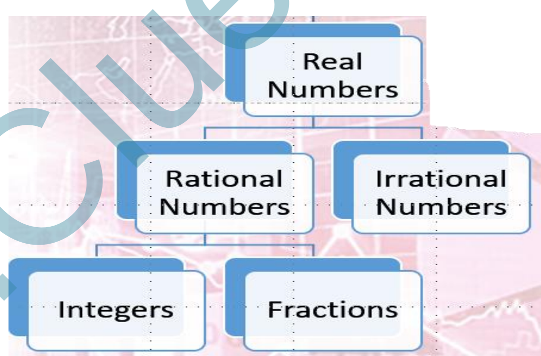
Irrational Numbers

- Other than rational
    - No repetitive pattern
    - Endless

  e.g. $\pi = (3.141592653589 \dots)$

Combining Rational & Irrational Numbers

- We get real numbers
- Integers & fractions need some description.



Integers vs Fractions

Integers

- Combination of zero, natural numbers, and –ve values of natural numbers.
- $(\dots, -3, -2, -1, 0, 1, 2, 3, \dots)$

Fractions

- Not a whole number (0, 1, 2, 3, …).

$^5/_6$

$^a/_b$

0.52

Lesson 03

USE OF SETS IN ECONOMICS

TOPIC 008: USE OF SETS IN ECONOMICS: SET NOTATION

Set Notation

- Collection of distinct objects.
- e.g. Numbers, labor, firms, countries etc.
- These objects are called 'elements' of sets
- Two ways to write sets: enumeration & description

Enumeration

B = {Al Baraka Bank Ltd., Barclays Bank PLC., Citibank, Deutsche Bank AG., Dubai Islamic Bank Ltd., Industrial and Commercial Bank of China Ltd., Standard Chartered Bank Ltd., The Bank of Tokyo-Mitsubishi UFJ Ltd}

Description

$$\boldsymbol{B = \{x \mid x\ a\ foreign\ bank\ in\ Pakistan\}}$$

- Read as: "B is the set of all (banks in Pakistan) x, such that x is a foreign bank in Pakistan."
- B is the set of all foreign banks in Pakistan.
- Membership of an element is denoted by $\in$
- (Al Baraka Bank Ltd.) $\in$ B

Range in Set Notation

Ranges can also be defined in set notation:

$$\boldsymbol{C = \{MPC \mid 0 \leq MPC \leq 1\}}$$

Relationship using Set Notation

If A = {Pakistan, India, Afghanistan, Bangladesh, Bhutan, Maldives, Nepal, Sri Lanka}
B = {Pakistan, Bangladesh, Afghanistan}

- B $\subset$ A (B is contained in A) or A$\supset$B (A includes B)
- Or B is a 'Subset' of A.

TOPIC 009: USE OF SETS IN ECONOMICS: OPERATIONS OF SETS

Set Operations

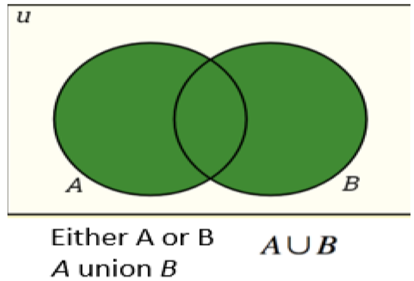
Three major operations:

- Union
- Intersection
- Complement

Union

- Similar to addition (+).
- Write all elements of all sets once only, regardless of their multiple presences in sets.
- Denoted by 'U'

Venn diagram of union



Green shaded area depicts union of set A and B

A= {1, 2, 3, 4, 5} & B = {1, 3, 5, 7}

Union is denoted by ∪.

A ∪ B = {1, 2, 3, 4, 5} U {1, 3, 5, 7}

A ∪ B = {1, 2, 3, 4, 5, 7}

In description, union of A and B:

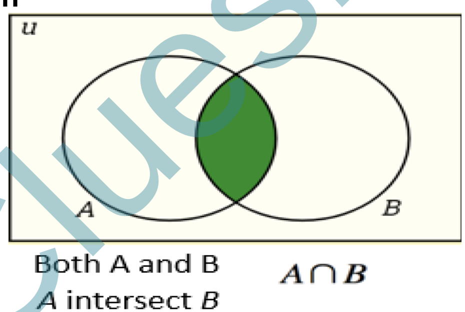
A U B = {x | x ∈ A or x ∈ B}

Intersection

- Write common elements of all sets.
- Resembles 'common factor' in factorization.
- Denoted by ∩
- **A ∩ B = {1, 3, 5}**
- In Description, Intersection of A and B:

A ∩ B = {x | x ∈ A and x ∈ B}

Venn Diagram of Intersection



Green shaded area depicts intersection of set A and B

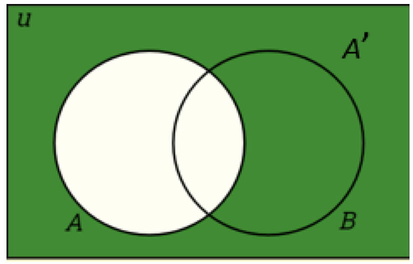
- Complement
- Write uncommon elements of the universal set ($\boldsymbol{U}$) in comparison to set under process ($\boldsymbol{A}$).
- Resembles 'subtraction'.
- Denoted by $\tilde{\boldsymbol{A}}$ or $\boldsymbol{A}'$
- Complement

If U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9},

A = {1, 2, 3, 4, 5}, then

- **$\tilde{A}$ or A'= U − A**
- **$\tilde{A}$ or A'= {0, 6, 7, 8, 9}**
- **$\tilde{A}$ or A'= {x | x ∈ U and x ∉ A}**

Venn Diagram of Complement



A' the complement of A

Green shaded area depicts complement of set A

TOPIC 010: USE OF SETS IN ECONOMICS: LAWS OF OPERATIONS OF SETS

Laws of Operations

Three major Laws:

- Commutative Law
- Associative Law
- Distributive Law

Commutative Law of Union

- Order does not matter.
- Of union: A U B = B U A

A= {1, 2, 3, 4, 5} & B = {1, 3, 5, 7}

A ∪ B = {1, 2, 3, 4, 5} U {1, 3, 5, 7}

A ∪ B = {1, 2, 3, 4, 5, 7}

B ∪ A = {1, 3, 5, 7} U {1, 2, 3, 4, 5}

B ∪ A = {1, 2, 3, 4, 5, 7}

Hence, **A U B = B U A**

Commutative Law of Intersection

A ∩ B = B ∩ A

A= {1, 2, 3, 4, 5} & B = {1, 3, 5, 7}

A ∩ B = {1, 2, 3, 4, 5} ∩ {1, 3, 5, 7}

A ∩ B = {1, 3, 5}

B ∩ A = {1, 3, 5, 7} ∩ {1, 2, 3, 4, 5}

B ∩ A = {1, 3, 5}

Hence, **A ∩ B = B ∩ A**

Associative Law of Union

- Order of 'selection' does not matter, while dealing with more than 2 sets.
- Of union:

A U (BUC)= (AUB) U C

A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}

- **L.H.S**

A U B = {1, 2, 3, 4, 5} U {1, 3, 5, 7}

A U B = {1, 2, 3, 4, 5, 7}

(A U B) U C = {1, 2, 3, 4, 5, 7} U {4, 6, 8, 10}

(A U B) U C = {1, 2, 3, 4, 5, 6, 7, 8, 10}

- **R.H.S**

B U C = {1, 3, 5, 7} U {4, 6, 8, 10}
B U C = {1, 3, 4, 5, 6, 7, 8, 10}
A U (B U C) = {1, 2, 3, 4, 5} U {1, 3, 4, 5, 6, 7, 8, 10}
A U (B U C) = {1, 2, 3, 4, 5, 6, 7, 8, 10}
Hence, **(A U B) U C = A U (B U C)**

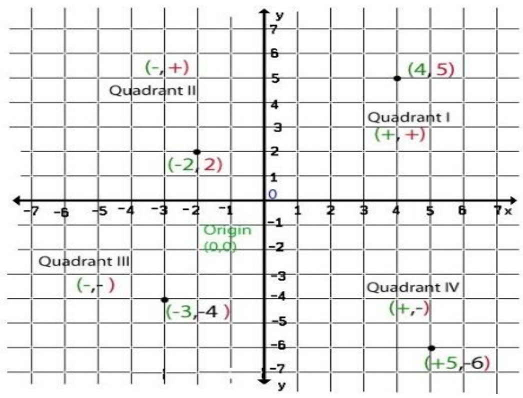
Associative Law of Intersection
A ∩ (B∩C) = (A∩B) ∩ C
A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}
L.H.S
B∩C = {1, 3, 5, 7} ∩ {4, 6, 8, 10}
B∩C = {}
A ∩ (B∩C) = {1, 2, 3, 4, 5} ∩ {}
A ∩ (B∩C) **= {} =** ∅ [theta]
R.H.S
A∩B = {1, 2, 3, 4, 5} ∩ {1, 3, 5, 7}
A∩B = {1, 3, 5}
(A∩B) ∩ C = {1, 3, 5} ∩ {4, 6, 8, 10}
(A∩B) ∩ **C = {} =** ∅ [theta]

Distributive Law of Union
A U (B∩C) = (AUB) ∩ (AUC)
A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}
L.H.S
(B∩C) = {}
A U (B∩C) = {1, 2, 3, 4, 5}
R.H.S
AUB = {1, 2, 3, 4, 5, 7}
AUC = {1, 2, 3, 4, 5, 6, 8, 10}
(AUB) ∩ **(AUC) = {1, 2, 3, 4, 5}**
Hence, **L.H.S = R.H.S**

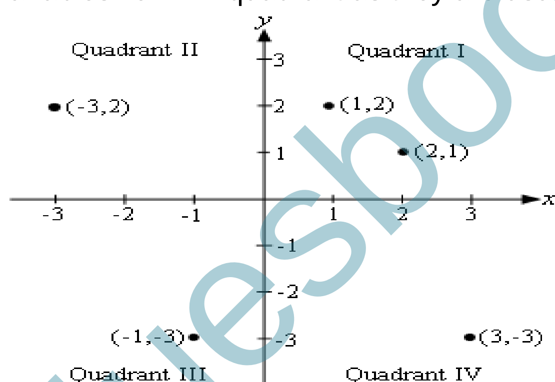
Distributive Law of Intersection
A ∩ (BUC) = (A∩B) U (A∩C)
A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}
L.H.S
(BUC) = {1, 2, 3, 4, 5, 6, 7, 8, 10}
A ∩ (BUC) = {1, 2, 3, 4, 5}
R.H.S
A∩B = {1, 3, 5}
A∩C = {2, 4}
(A∩B) U (A∩C) = {1, 2, 3, 4, 5}
Hence, **L.H.S = R.H.S**

TOPIC 011: CARTESIAN COORDINATES

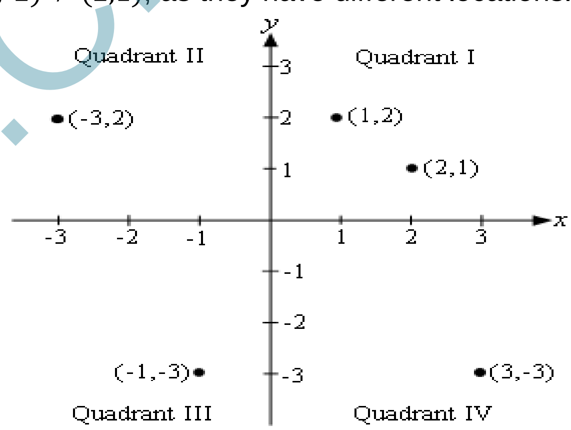
- Named after d*escartes* (French philosopher, mathematician, and scientist).
- Intersection of 2 numbers lines ($X' \leftrightarrow X$ and $Y' \leftrightarrow Y$) horizontally and vertically, respectively.



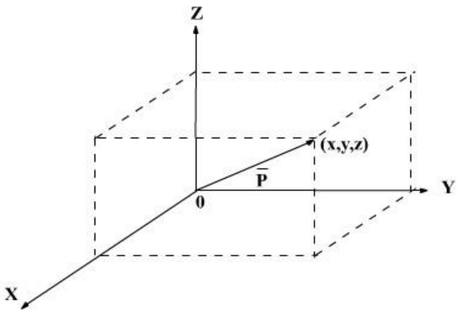
- 4 slices called quadrants (quarters of a circle).
- Ordered pairs are two variables (X & Y), in order, within parentheses:
  - 1st quadrant: (+ve, +ve)
  - 2nd quadrant: (-ve, +ve)
  - 3rd quadrant: (-ve, -ve)
  - 4th quadrant: (+ve, -ve)
- Most of economic variables lie in 1st quadrant as they are usually non-negative (≥ 0).

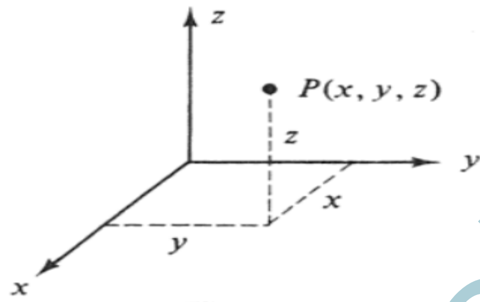


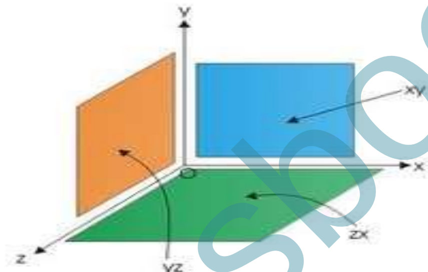
- Order in the coordinates is not interchangeable$(x,\ y) \neq (y, x)$.
- In quadrant – I, $(1,\ 2) \neq (2,1)$, as they have different locations.



- For 3-dimensional (3D) graphs, ordered triples are used $(x,\ y, z)$.
- Instead of lines, surfaces are generated.
- e.g. A cube is 3D.


Z
(x,y,z)
P
0
Y
X


z
P(x, y, z)
z
y
x
y
x

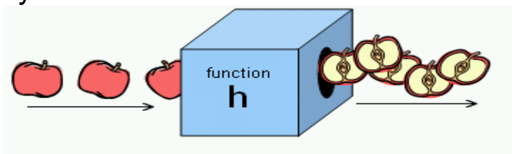

y
xy
x
zx
z
yz

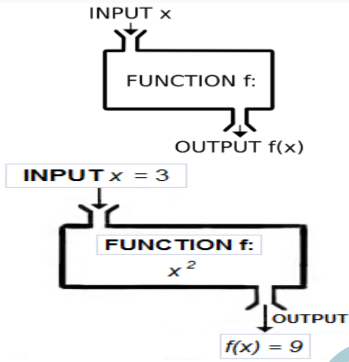
Lesson 04

USE OF FUNCTIONS IN ECONOMICS

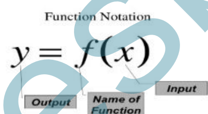
TOPIC 012: WHAT ARE FUNCTIONS?

"Power to act in a specific way."





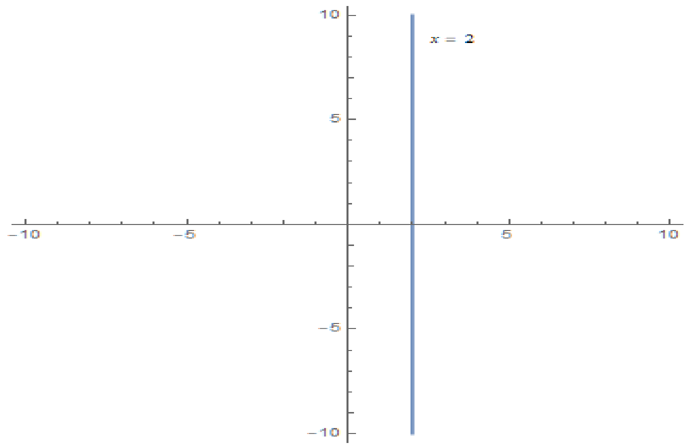
- System to write the dependence of one variable (x) on other (y).
- y is dependent variable.
- x is independent variable.



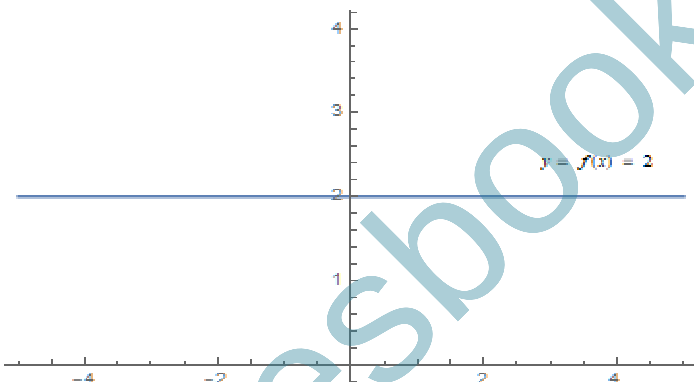
- In addition to *f*, F, G, the Greek letters φ (phi) and ψ (psi), and their capitals, are used to show functions.
- Sometimes, the dependent variable is itself used instead of '*f*'.
  y = y(x), z = z(x) etc.
- It is suitable to use different symbols for multiple functions with same independent variable.
  y=f(x), z = g(x) etc.

Plotting a Function

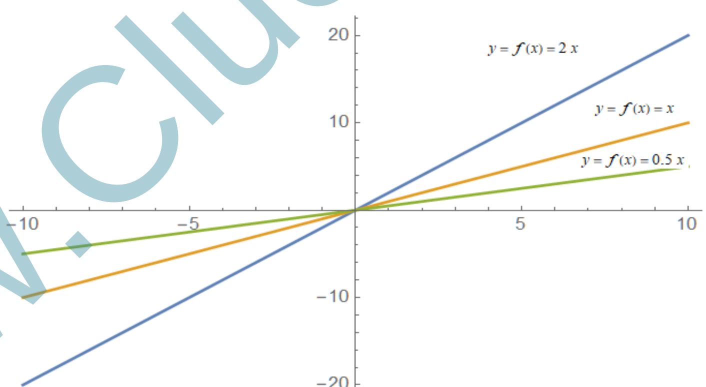
Function (x=2) depicts a vertical line parallel to y-axis.



- Function; y=f(x)=2 portrays a horizontal line parallel to x-axis.



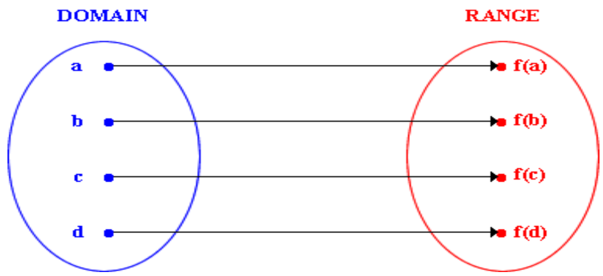
- Functions; y=f(x)=2x, y=f(x)=x & y=f(x)=0.5x.



- Plotting a function gives a locus (route/way) of points – hence called mapping.
- Function also converts values of independent variable into that of dependent variable – hence called transformation.

TOPIC 013: DOMAIN AND RANGE IN A FUNCTION

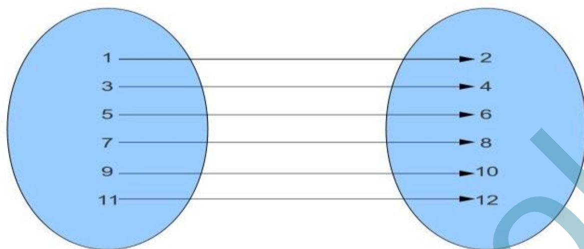
- All permissible values of x are 'Domain'
- All permissible values of y are 'Range'.



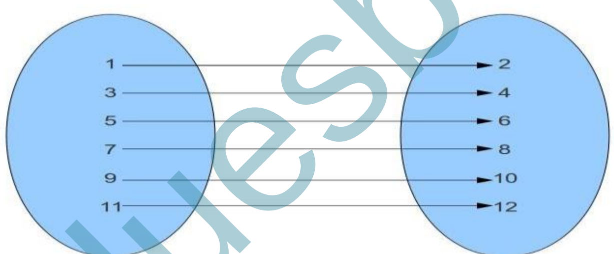
- Numerically speaking:

$$\{(1,2),(3,4),(5,6),(7,8),(9,10),(11,12)\}$$

- Ordered pairs show the mapping of function.



Numerically speaking:
- Domain={1,3,5,7,9,11}
- Range={2,4,6,8,10,12}

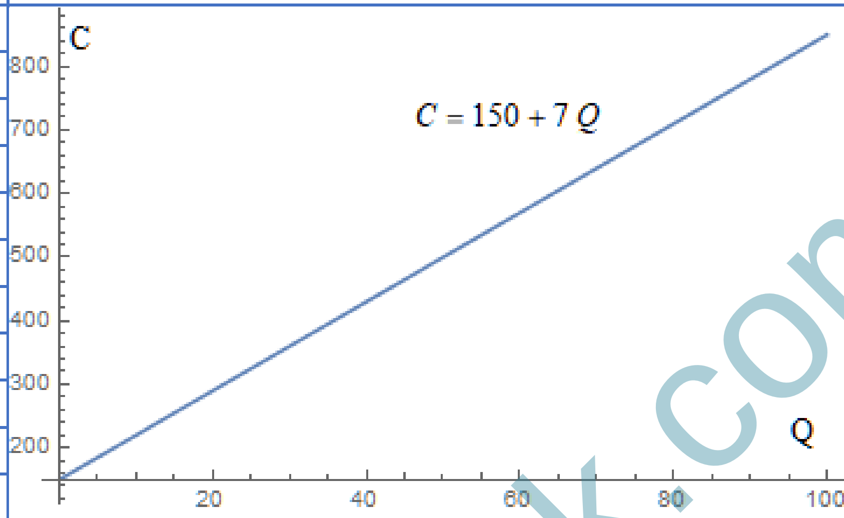


Economic Example
- Total cost/day 'C' of a firm depends on its daily output
- Its functional form is:

C = 150 + 7Q.
- Capacity limit = 100 units/day.

$\boldsymbol{Domain} = \{\boldsymbol{Q}|\boldsymbol{0} \leq \boldsymbol{Q} \leq \boldsymbol{100}\}$
$\boldsymbol{Range} = \{\boldsymbol{C}|\boldsymbol{150} \leq \boldsymbol{C} \leq \boldsymbol{850}\}$
- Extreme values ($\boldsymbol{Q} = \boldsymbol{0}$ & $\boldsymbol{C} = \boldsymbol{150}$) don't necessarily occur, however.

Daily Output	Cost/Day
0	150
10	220
20	290
30	360
40	430
50	500
60	570
70	640
80	710
90	780
100	850



TOPIC 014: DIFFERENCE BETWEEN FUNCTIONS AND RELATIONS

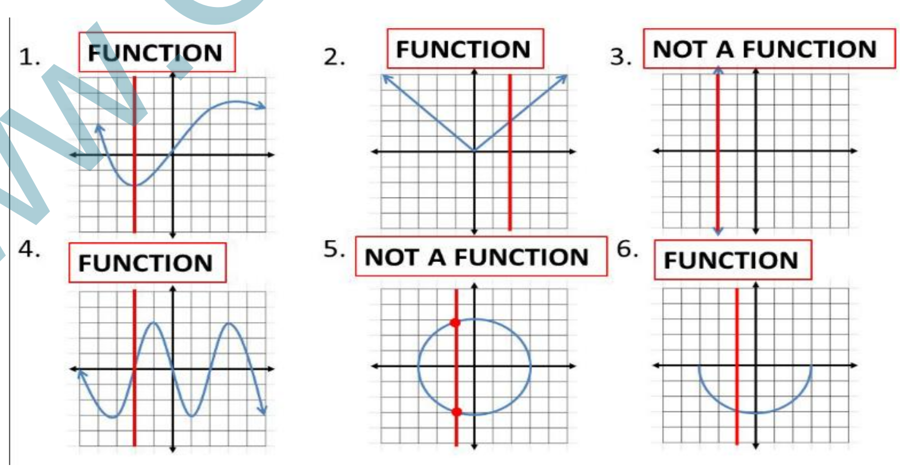
Slight but noteworthy difference.
Formation of ordered pairs

Function: (1,2),(2,4),(3,6)
Relation: (**1**,**3**),(2,5),(**1**,**7**).

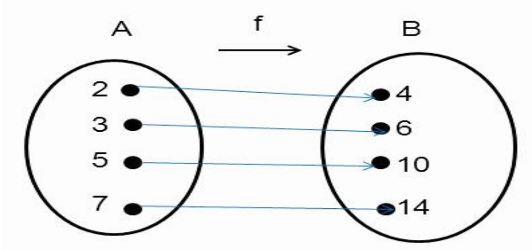
- One value of dependent variable (**1**) has multiple corresponding values of independent variable (**3**, **7**).
- Formerly a.k.a. single-valued function and multi-valued function, respectively.

Vertical Line Test
Function passes vertical line test while **relation** does not.

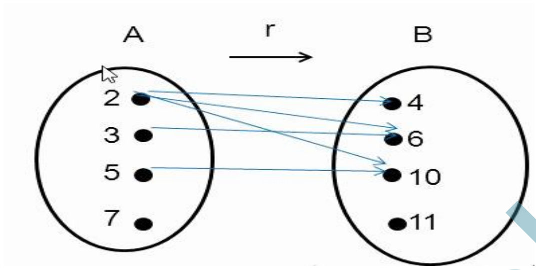
- A vertical line cuts a function on one point only.
- While it cuts a relation on multiple points.
- Case no. 5 is classic example of relation.



Mapping of Function (Formerly)



Mapping of Relation



TOPIC 015: ECONOMIC EXAMPLE OF SCHULTZ DEMAND FUNCTION

- H. Schultz estimated US cotton demand for the period 1915–1919.
- $\boldsymbol{D(P) = 6.4 - 0.3P}$
- If $\boldsymbol{P = 8}$ then

$D(8) = 6.4 - 0.3(8)$

$\boldsymbol{D(8) = 4}$ $Units$

- If $\boldsymbol{P = 10}$ then

$D(10) = 6.4 - 0.3(10)$

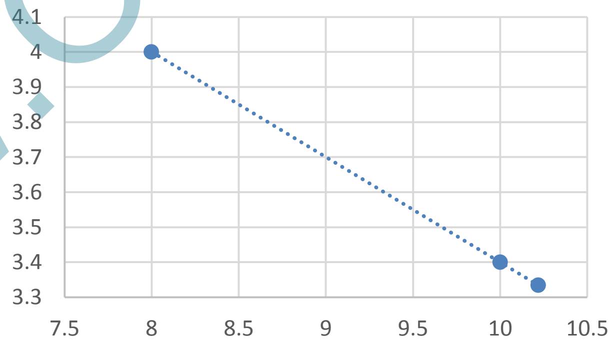
$\boldsymbol{D(10) = 3.4}$ $Units$

- If $\boldsymbol{P = 10.22}$ then

$D(10.22) = 6.4 - 0.3(10.22)$

$\boldsymbol{D(10.22) = 3.334}$ $Units$

- As expected, negatively sloped demand curve is generated.



- Conversely, we can find price of cotton ($\boldsymbol{P}$), if quantity demanded ($\boldsymbol{D}$) is given:
- $\boldsymbol{D = 6.4 - 0.3P}$
- If $\boldsymbol{D = 3.13}$ then

$3.13 = 6.4 - 0.3(P)$

$\boldsymbol{P = \frac{3.27}{0.3}}$

$\boldsymbol{P = 10.9}$ $Units$

TOPIC 016: ECONOMIC EXAMPLE OF COST FUNCTION OF CLEANING IMPURITIES FROM A LAKE:

Cleaning cost of p% of impurities in a lake:

$$\text{b(p)} = \frac{10p}{105 - p}$$

- If $\boldsymbol{p} = \mathbf{0}$ then

$b(0) = \frac{10(0)}{105-0}$

$\boldsymbol{b}(\mathbf{0}) = \mathbf{0}\ Units$

- If $\boldsymbol{p} = \mathbf{50}\%$ then

$b(50\%) = \frac{10(50)}{105-50}$

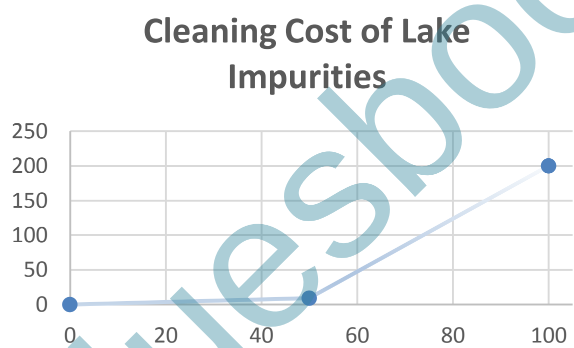
$\boldsymbol{b}(\mathbf{50}\%) = \mathbf{9.09}\ Units$

- If $\boldsymbol{p} = \mathbf{100}\%$ then

$b(100\%) = \frac{10(100)}{105-100}$

$\boldsymbol{b}(\mathbf{100}\%) = \mathbf{200}\ Units$

- The increase in cleaning cost of lake impurities has an increasing trend.
- However, it does not grow as a straight line.



Additional cost of additional cleaning (h%) above p% can be written as follows:

$$= \text{b(p + h)} - \text{b(p)} = \frac{10(p+h)}{105-(p+h)} - \frac{10p}{105-p}$$

If $\boldsymbol{p} = \mathbf{50}\%$, $\boldsymbol{h} = \mathbf{30}\%$ then

$$= \frac{\mathbf{10(50\% + 30\%)}}{\mathbf{105 - (50\% + 30\%)}} - \frac{\mathbf{10(50\%)}}{\mathbf{105 - (50\%)}}$$

$$= 32 - 9.09$$

$$= \mathbf{22.09}$$

Additional cost of cleaning above 50% is 22.09 Units.

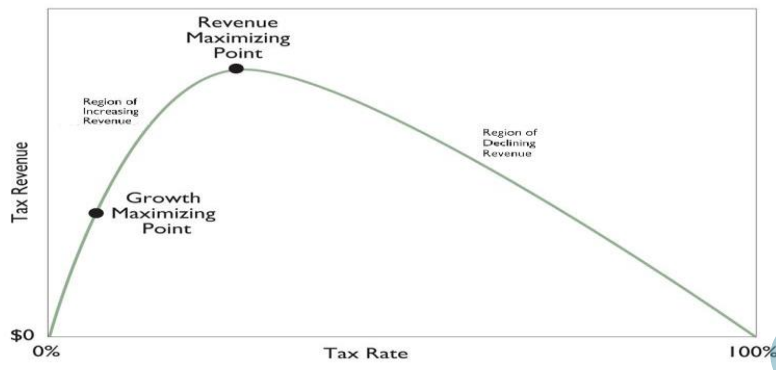
TOPIC 017: ECONOMIC EXAMPLE OF FUNCTION: LAFFER CURVE

Laffer curve shows the theoretical relationship between rates of taxation and the corresponding levels of government revenue.

- In functional form:

$$\textbf{Government Revenue} = \boldsymbol{f}(\boldsymbol{Tax\ Rates})$$

The Laffer Curve



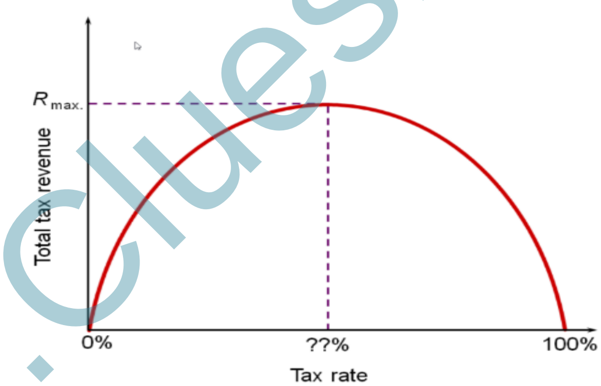
- Till the point of 'revenue maximization', the relationship between tax rates and tax revenue is positive/direct.

$$\boldsymbol{Tax\ Revenue} \propto \boldsymbol{Tax\ Rates}$$

- After this point, the relationship becomes negative/inverse.

$$\boldsymbol{Tax\ Revenue} \propto \frac{\mathbf{1}}{\boldsymbol{Tax\ Rates}}$$

- The tax revenue maximizing point in the Laffer curve can be found using *Rolle's* theorem.
- It is a calculus-based theorem but is beyond the scope of this course.



Lesson 05

CONSTANT FUNCTIONS AND LINEAR FUNCTIONS

TOPIC 018: TYPES OF FUNCTIONS: CONSTANT FUNCTIONS

Functions where the power of independent variable is zero (0).

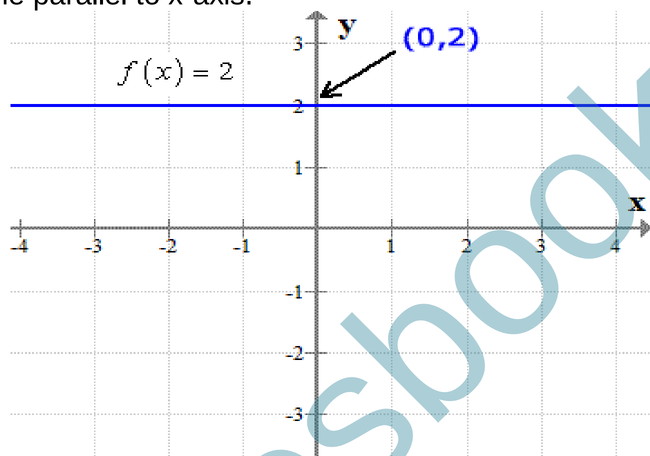
e.g. $y = f(x) = ax^0$

$$= a(1) = a$$

Where, 'a'' can be any constant value.

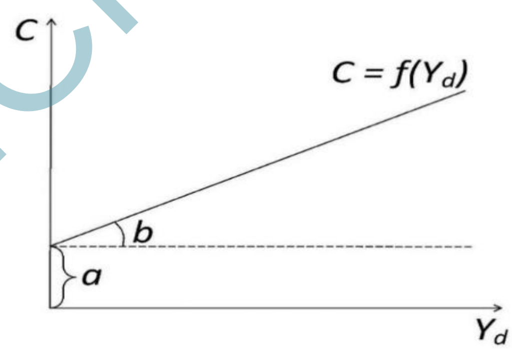
- The graphs of such constant functions appear as horizontal lines.

- In constant function, $y = a$, let a = 2.
- Then, $y = f(x) = 2$.
- Its graph is a line parallel to x-axis.



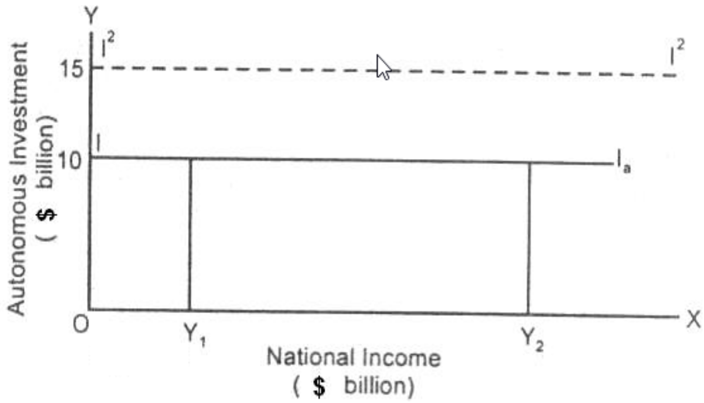
Autonomous Consumption (C_o)

- Consumption at $Y_d = 0$.
- $C_o \neq f(Y_d)$.
- Consumption independent of disposable income.



Autonomous Investment (I_o)

- Investment at $Y = 0$.
- $I_o \neq f(Y)$.



Autonomous Government Spending (G_o)

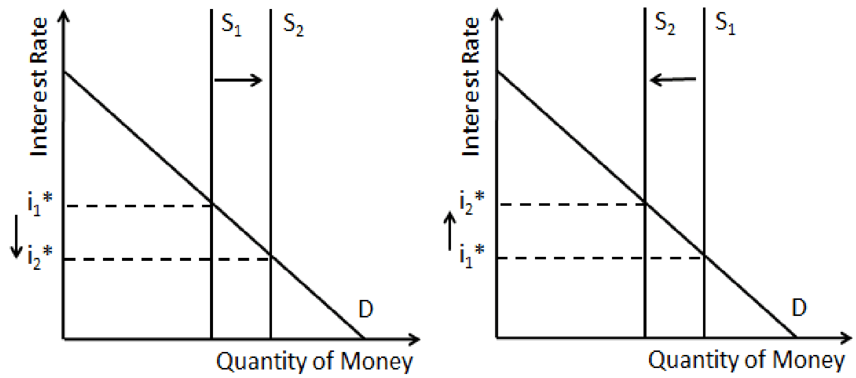
- Government spending is usually based on political will (independent of Y)
- $G_o \neq f(Y)$.

Autonomous Tax (T_o)

- Taxes independent of level of income (Y).
- e.g. Lump-sum taxes.
- $T_o \neq f(Y)$.

Supply of Money (M_o)

- In the short run, supply of money is usually independent of interest rate (i).
- Determined by central monetary authority (e.g. SBP, FED etc.).
- $M_s = M_o \neq f(i)$.



TOPIC 019: TYPES OF FUNCTIONS: POLYNOMIAL FUNCTION: LINEAR FUNCTIONS

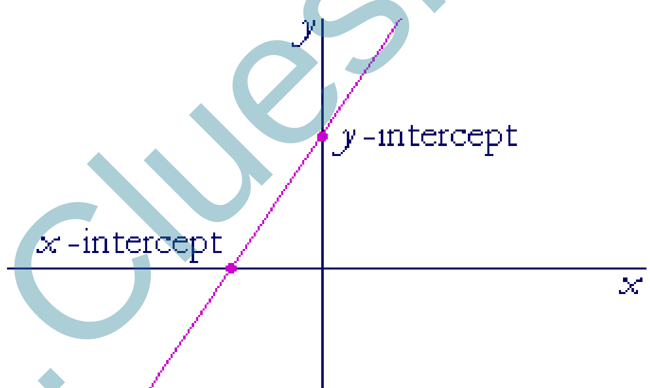
Polynomial Functions

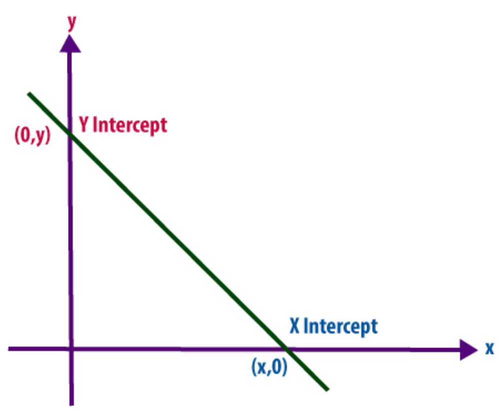
Etymology: {('poly' (many) + 'nomial' (parts)}.

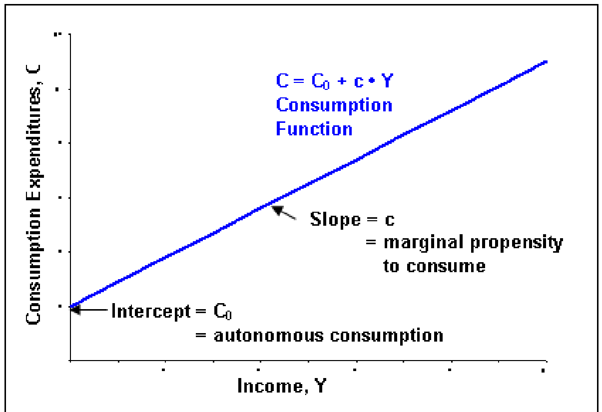
$$y = a_o + a_1x + a_2x^2 + \cdots + a_nx^n$$

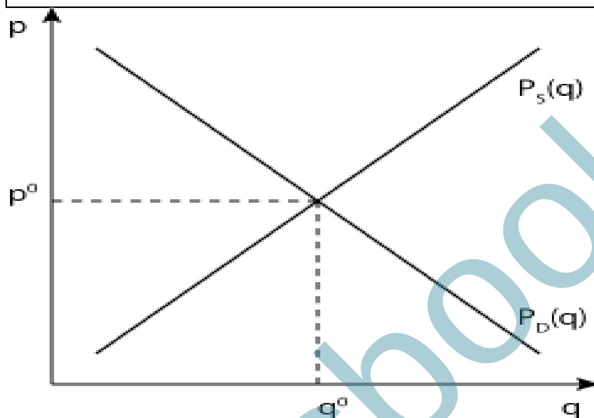
Linear Functions

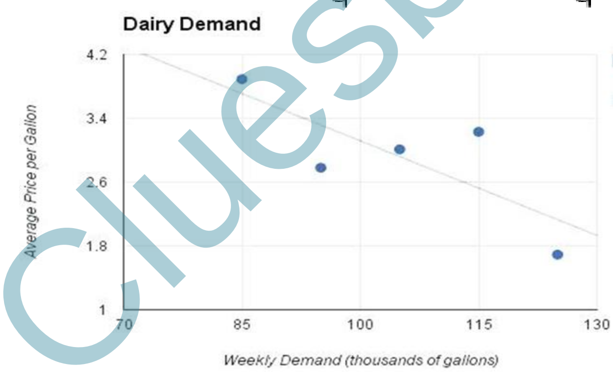
- Different values of n give different types of functions.
- Linear function $(n = 1) \Rightarrow y = a_o + a_1x$
- Degree of linear equation = 1.
- If $x = 0$, then $y = a_o$, which gives a y-intercept (vertical intercept).
- It holds true for both positive and negative sloped linear function.
- If $a_1 > 0$, +ve slope.
- If $a_1 < 0$, -ve slope.











TOPIC 020: INTERPRETING LINEAR ECONOMIC FUNCTIONS

Estimated cost function for the US Steel Corp. (1917–1938).

$$C = 55.73x + 182,100,000$$

Interpreting linear Function

Compare with slope intercept form of a straight line:

$$y = m.x + c$$

Where, $m = slope$ **&** $c = intercept.$

- Here, slope = 55.73,
- If production increases by 1 ton, then the cost increases by $55.73.
- Estimated annual demand function for rice in India for the period 1949–1964.

$$Q = -0.15P + 0.14$$

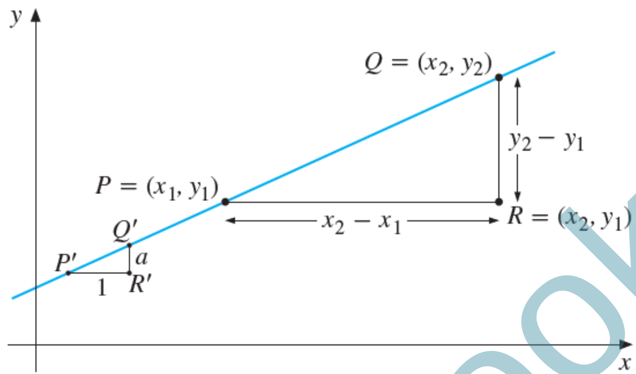
Interpretation

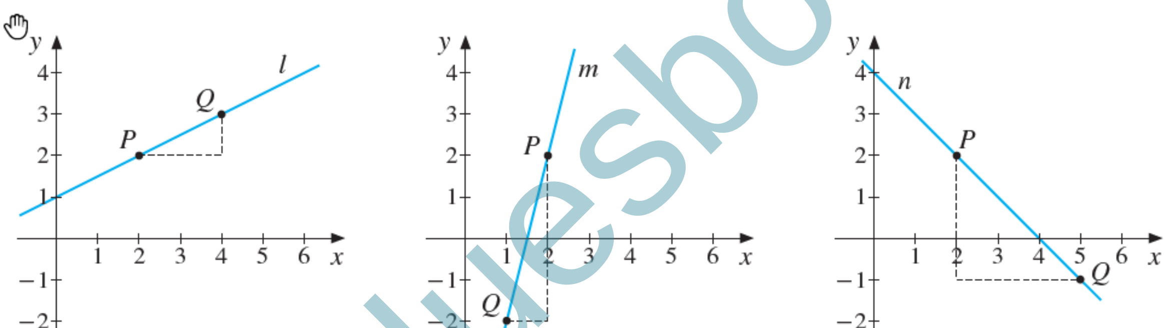
- Similar to last example.
- The slope is −0.15.

- If price increases by one Indian rupee, then the quantity demanded decreases by 0.15 units.

Calculating Slope of a Straight Line

- Take two points e.g. $\boldsymbol{P(x_1, y_1)}$ & $\boldsymbol{Q(x_2, y_2)}$.
- Draw perpendicular from higher point and horizontal line from lower point to get a triangle ΔPRQ.
- $\boldsymbol{Slope} = \frac{\boldsymbol{Rise}}{\boldsymbol{Run}} = \frac{\boldsymbol{Altitude}}{\boldsymbol{Base}} = \frac{y_2 - y_1}{x_2 - x_1}$ $where\ (\boldsymbol{x_2 \neq x_1})$





$$a_l = \frac{3-2}{4-2} = \frac{1}{2}, \quad a_m = \frac{-2-2}{1-2} = 4, \quad a_n = \frac{-1-2}{5-2} = -1$$

<u>TOPIC 021: APPLICATIONS OF LINEAR FUNCTIONS: POPULATION AND CONSUMPTION FUNCTIONS</u>

Population Function

European population was 641 million in 1960, and 705 million in 1970.

- Let $\boldsymbol{P = Population}$ (in millions) & $\boldsymbol{t = time}$ (in years).
- $\boldsymbol{t = 0}$ for 1960 & $\boldsymbol{t = 1}$ for 1961 and so on.
- Linear function:

$$\boldsymbol{P = a.t + b}$$

- Given points:$(t_1, P_1) = (0,641)$ and $(t_2, P_2) = (10,705)$.
- Using point-point formula of slope of straight line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$P - 641 = \frac{705 - 641}{10 - 0}(t - 0) = \frac{64}{10}t$$

$P = 6.4\,t + 641$

- Back-cast and forecast

Table Population estimates for Europe

Year	1930	1975	2000
t	−30	15	40
Formula	449	737	897

Consumption Function

Haavelmo estimated for US economy (1929–1941):

$\boldsymbol{C = 95.05 + 0.712(Y)}$

$\boldsymbol{C = C_o + MPC\,.\,Y}$

$\boldsymbol{C_o}$ is autonomous consumption.

- Geometrically, it shows the intercept of the linear function.

$\boldsymbol{MPC = 0.712}$.

- About 71.2% of increase in income was being spent in US.
- ***MPC*** also shows the slope of consumption function.

Lesson 06

QUADRATIC FUNCTIONS AND CUBIC FUNCTIONS

TOPIC 022: TYPES OF FUNCTIONS: POLYNOMIAL FUNCTION: QUADRATIC FUNCTIONS

Polynomial Functions

$$y = a_o + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Quadratic Functions

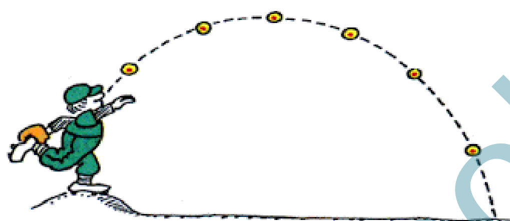
$$(n = 2)$$

$$\Rightarrow y = a_o + a_1 x + a_2 x^2$$

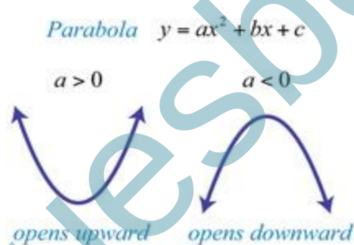
Or standard form of quadratic equation is:

$$y = ax^2 + bx + c$$

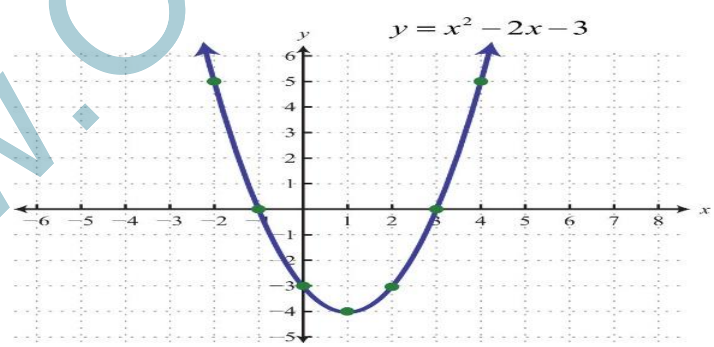
- Degree of quadratic equation = 2.
- Gives a 'parabola'

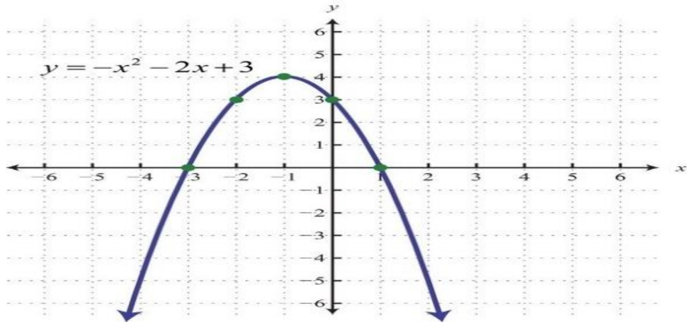


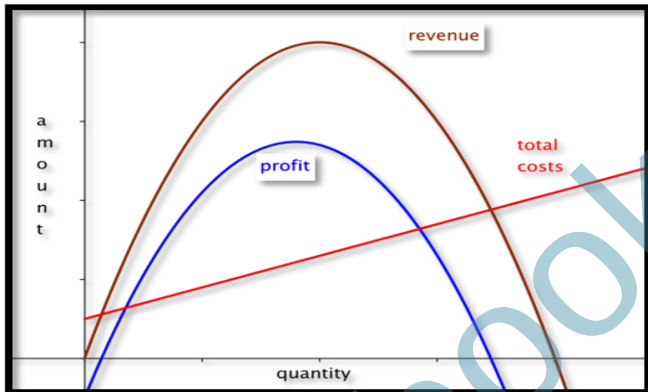
- **Etymology:** {*para* 'beside' + *bolē* 'a throw'}.
- Curve with **a single** bump or wiggle either in a valley or hill.

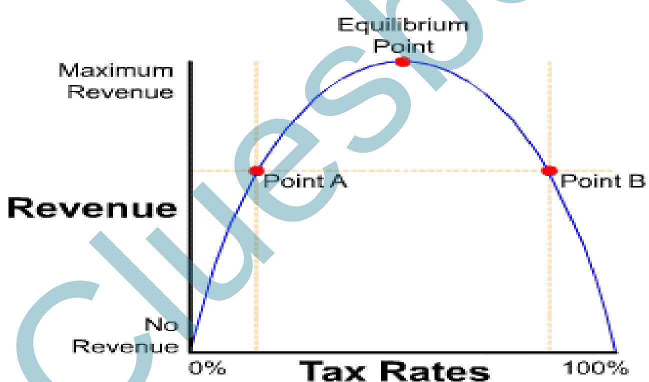


- First graph shows a happy/U-shaped/open upwards parabola (valley), as $a = (1) > 0$.
- Whereas, second is sad/Inverted U-shaped/open downwards parabola (hill), as $a = (-1) < 0$.

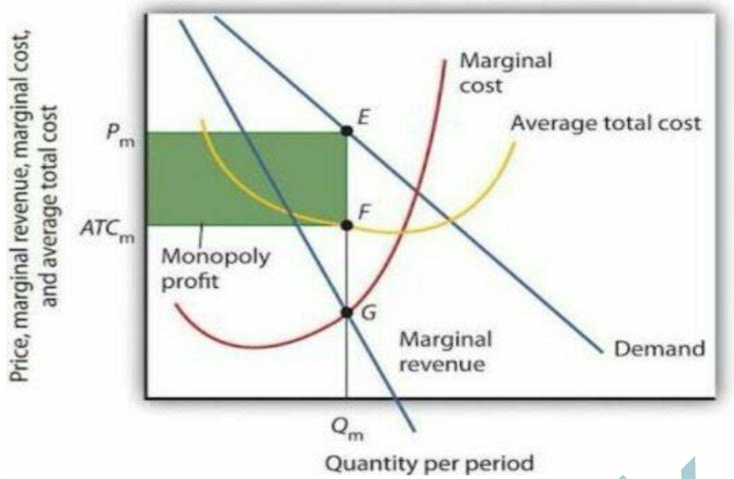



$y = -x^2 - 2x + 3$


revenue
profit
total costs
amount
quantity


Equilibrium Point
Maximum Revenue
Point A
Point B
Revenue
No Revenue
0%
Tax Rates
100%

TOPIC 023: QUADRATIC COST FUNCTION AND PROFIT FUNCTION OF A MONOPOLY



Quadratic Cost Function

$$C = 2Q + \frac{1}{2}Q^2$$
$$P = 102 - 2Q$$
$$\pi(Q) = PQ - C$$
$$= (102 - 2Q)Q - \left(2Q + \frac{1}{2}Q^2\right)$$
$$= 100Q - \frac{5}{2}Q^2$$

Condition for maximization of quadratic function/at parabola

If $a < 0$, then $f(x) = ax^2 + bx + c$ has its **maximum** at $x = -b/2a$

$$Q^* = -\frac{100}{2(-\frac{5}{2})} = 20$$

$$\pi^* = \pi(Q^*) = 1000$$

Quadratic Profit Function

$$C = \alpha Q + \beta Q^2, \qquad Q \geq 0$$

$$P = a - bQ, \qquad Q \geq 0$$

$$\pi(Q) = R - C = (a - bQ)Q - \alpha Q - \beta Q^2 = (a - \alpha)Q - (b + \beta)Q^2$$

Condition for maximization of quadratic function/at parabola

If $a < 0$, then $f(x) = ax^2 + bx + c$ has its **maximum** at $x = -b/2a$

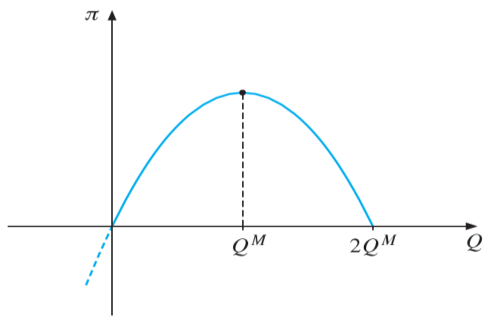
- Here, for quadratic formula:

$$a = -(b + \beta)$$

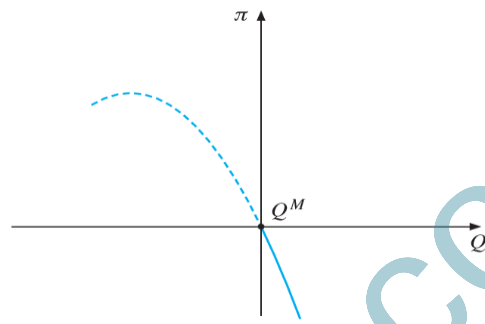
$$b = (a - \alpha)$$

$$Q^M = \frac{a - \alpha}{2(b + \beta)} \quad \text{with} \quad \pi^M = \frac{(a - \alpha)^2}{4(b + \beta)}$$

$$Q^M = \frac{a - \alpha}{2(b + \beta)} \quad \text{with} \quad \pi^M = \frac{(a - \alpha)^2}{4(b + \beta)}$$



The profit function, $a > \alpha$



The profit function, $a \leq \alpha$

TOPIC 024: QUADRATIC FUNCTION AND PRODUCTION POSSIBILITIES FRONTIER

$$\boldsymbol{y = -\frac{1}{4}x^2 - \frac{1}{2}x + 42}$$

Where, y and x are two goods.
Let $y = 0$;

$$\boldsymbol{-\frac{1}{4}x^2 - \frac{1}{2}x + 42 = 0}$$

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\boldsymbol{x = 12 \text{ or } x = -14}$$

$$\boldsymbol{x - intercepts = \{(12, 0)(-14, 0)\}}$$

$$\boldsymbol{y = -\frac{1}{4}x^2 - \frac{1}{2}x + 42}$$

Now, **let $x = 0$;**

$$\boldsymbol{y = -\frac{1}{4}(0)^2 - \frac{1}{2}(0) + 42}$$

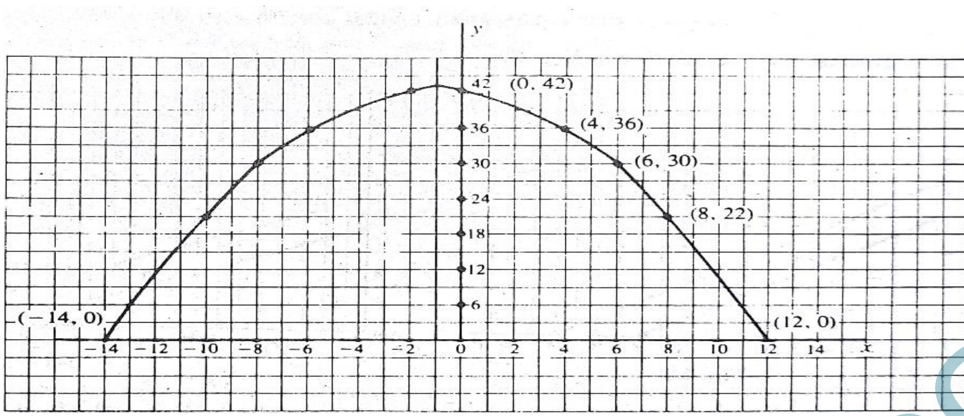
$$\boldsymbol{y = 42}$$

$$\boldsymbol{y - intercept = \{(0, 42)\}}$$

Combining the 3 intercepts;

$$\boldsymbol{\{(12, 0), (-14, 0), (0, 42)\}}$$

- Only intercepts cannot give a precise production possibilities frontier.
- Other points on curve are also required where both coordinates are non-zero.



TOPIC 025: TYPES OF FUNCTIONS: POLYNOMIAL FUNCTION: CUBIC FUNCTIONS

Cubic Functions

$$y = a_o + a_1x + a_2x^2 + \cdots + a_nx^n$$

Cubic Functions

$$(n = 3)$$

$$\Rightarrow y = a_o + a_1x + a_2x^2 + a_3x^3$$

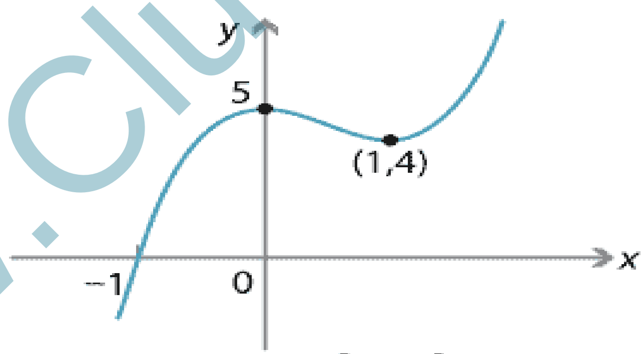
Or standard form of cubic equation is:

$$y = ax^3 + bx^2 + cx + d$$

a, b, c & d are constants

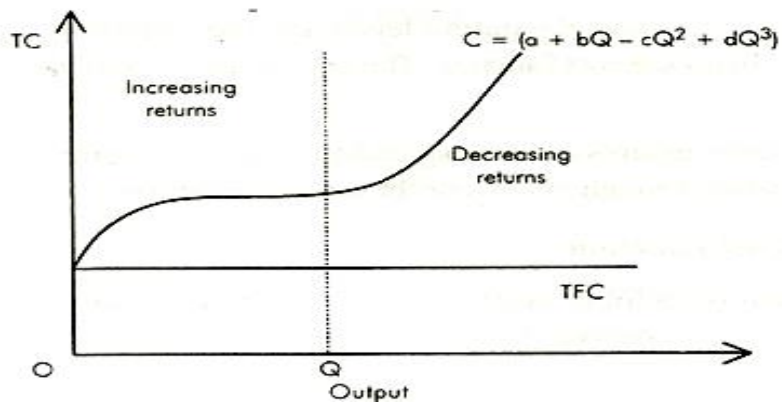
- $a \neq 0$, as ax^3 is the requisite for a cubic function
- Degree of cubic equation = 3.

$$y = f(x) = 2x^3 - 3x^2 + 5$$



Graph of $f(x) = 2x^3 - 3x^2 + 5$.

- Curve with **two** bumps or wiggles.



TOPIC 026: CUBIC COST FUNCTIONS

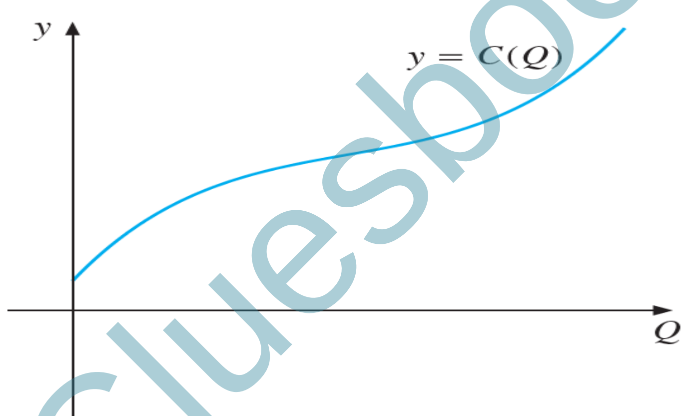
Cubic Cost Functions

$C(Q) = aQ^3 + bQ^2 + cQ + d$

$a > 0, b < 0, \quad c > 0, d > 0,$

- Then

$3ac > b^2$



A cubic cost function

Example

$\boldsymbol{C(Q) = 2Q^3 - 3Q^2 + 4Q + 5}$

- Then $\boldsymbol{3ac > b^2}$ should hold.
- Here $\boldsymbol{a = 2, b = -3,\ c = 4}$ & $\boldsymbol{d = 5}$.

$\boldsymbol{= 3(2)(4) > (-3)^2}$

$\boldsymbol{= 24 > 9}$

Which is true.

Electric Power Generating Plant Cubic Cost Function

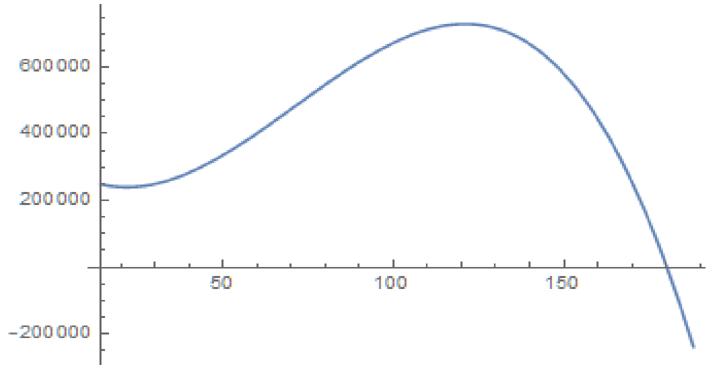
$$\boldsymbol{C(Q) = -Q^3 + 214.2Q^2 - 7900Q + 320700}$$

Here $\boldsymbol{a = -1 \ngtr 0, b = 214.2 \nless 0,\ c = -7900 \ngtr 0}$, & $\boldsymbol{d = 320700} > 0$.

- Testing the validity of $\boldsymbol{3ac > b^2}$.

$$\boldsymbol{= 3(-1)(-7900) > (214.2)^2}$$

$\boldsymbol{= 23700 \ngtr 45881.64}$, Which is against the condition.



Components of Cost Function (FC & VC)

$C(Q) = -Q^3 + 214.2Q^2 - 7900Q + 320700$

Here, $FC = 320700$.

$VC(Q) = -Q^3 + 214.2Q^2 - 7900Q$

Lesson 07

RATIONAL FUNCTIONS AND EXPONENTIAL FUNCTIONS

TOPIC 027: RATIONAL FUNCTIONS

Rational Functions

Ratio of two polynomial functions.

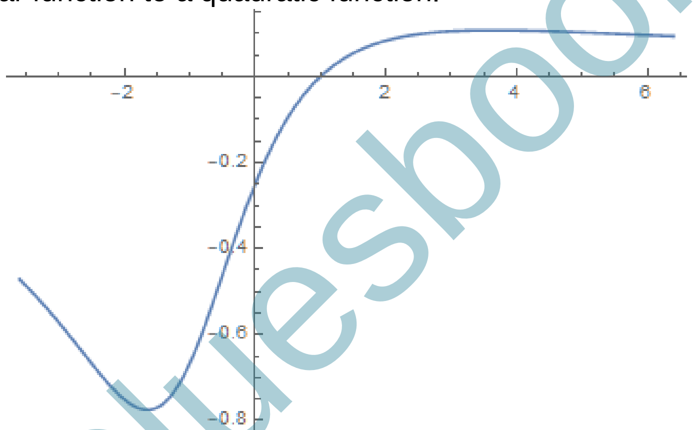
$$y(x) = \frac{a_o + a_1x + a_2x^2 + \cdots + a_nx^n}{a_o + a_1x + a_2x^2 + \cdots + a_mx^m}$$

- n and m are not necessarily same.
- It shows that it is not necessary to have same degree of equation in numerator and denominator.
- $a_o + a_1x + a_2x^2 + \cdots + a_mx^m \neq 0$, else the rational function will become undefined.

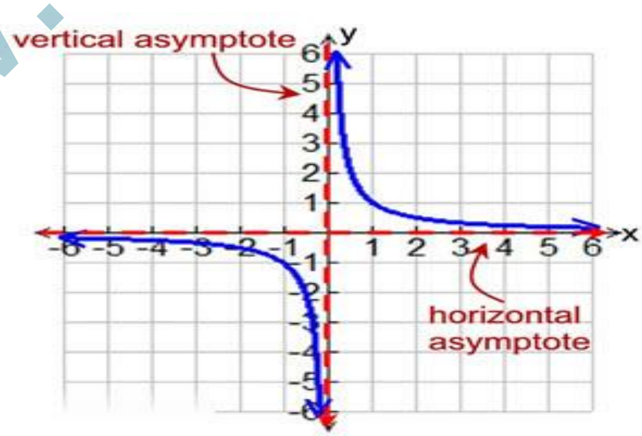
$$y(x) = \frac{x-1}{x^2+2x+4}$$

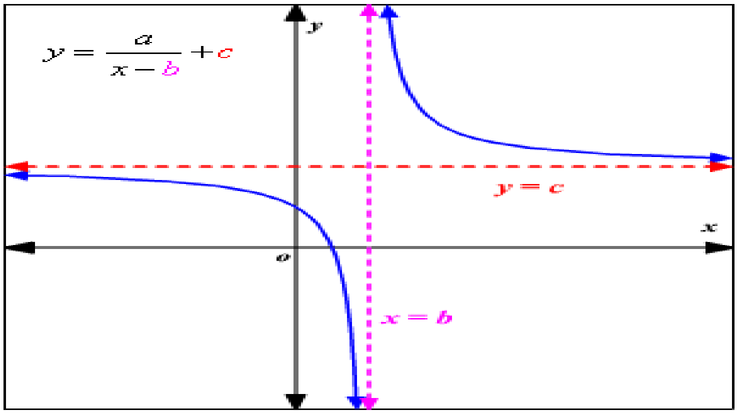
$x^2 + 2x + 4 \neq 0$, for $y(x)$ to be defined.

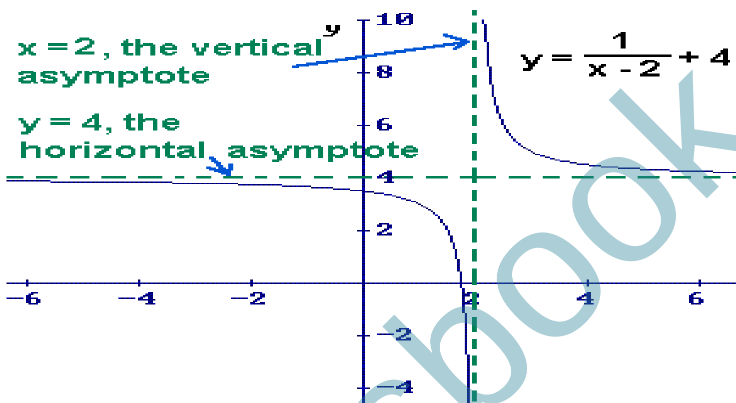
- Here n = 1 and m = 2.
- Ratio of a linear function to a quadratic function.



Asymptote is a line or curve that approaches a given curve arbitrarily closely.
Vertical asymptote is at such value of **x** that turns **y** into infinity.
Horizontal asymptote is at such value of **y** that turns **x** into infinity.





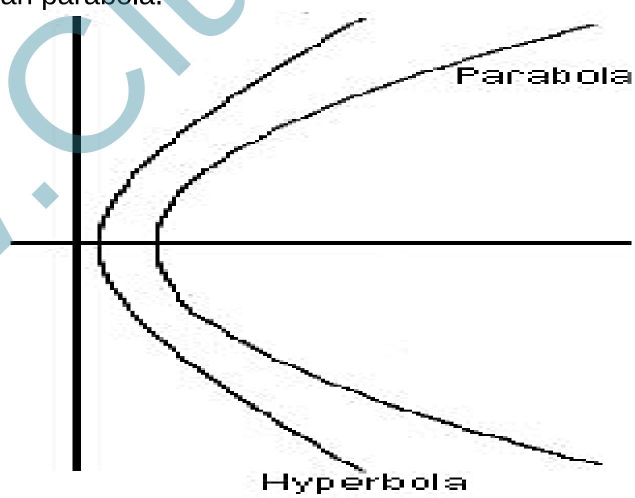


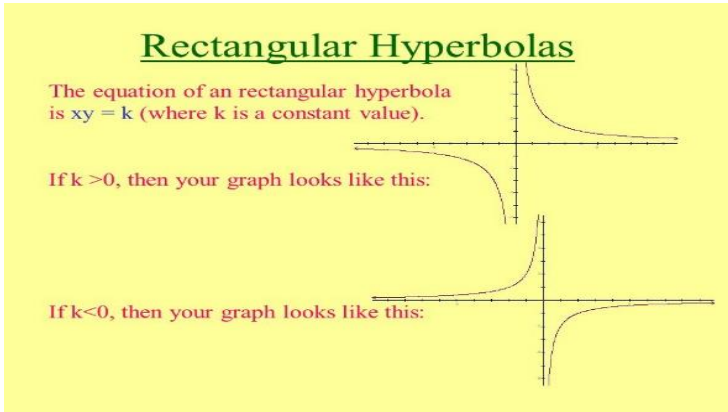
TOPIC 028: OTHER TYPES: RECTANGULAR HYPERBOLIC FUNCTIONS

Rectangular Hyperbolic Functions

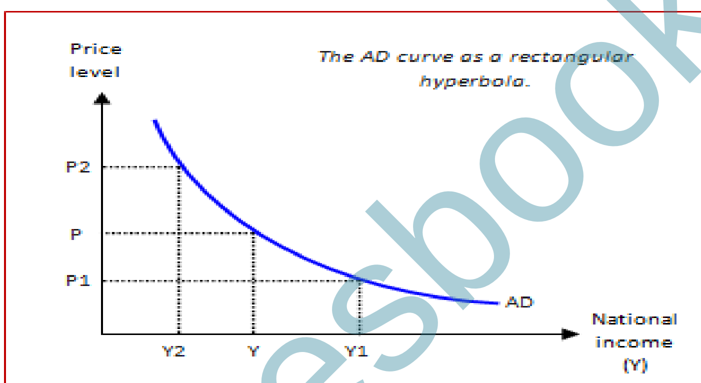
Hyperbola: Etymology: {Hyper (Beyond)+Bola (Throw)}.

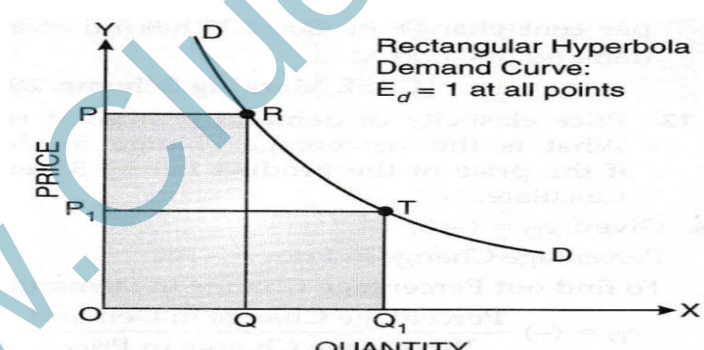
- Similar to parabola but not the same.
- Usually wider than parabola.

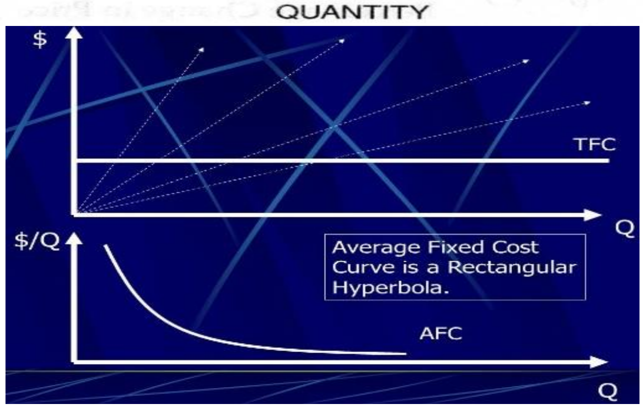




- Rectangular hyperbolic functions have x and y asymptotes.
- Economic examples include aggregate demand (AD) curve.







TOPIC 029: OTHER TYPES: NON-ALGEBRAIC EXPONENTIAL FUNCTION

Etymology: Latin origin; *Expōnēre*. In english expound: explain in detail.

- Independent variable occurs as a root or power.

$f(x) = Aa^{x}$ where, A>0 and $a > 0$.

- Non-linear graphs.

- For $x = 0$,

$$f(0) = Aa^{\mathbf{0}}$$
$$f(0) = A$$

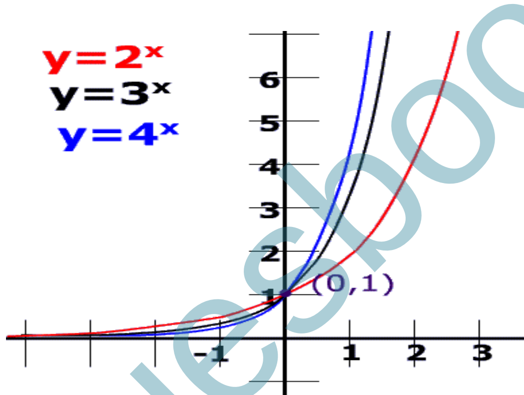
- Exponential function reduces to a constant function if $x = 0$.
- Caveat: unlike power functions which have variables in base instead of exponent.

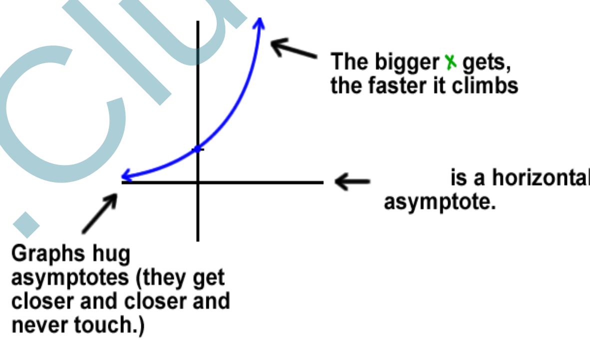
e.g. $f(x) = A\boldsymbol{x}^{a}$

- Exponential function can either have growth or decay.

Case of Exponential Growth

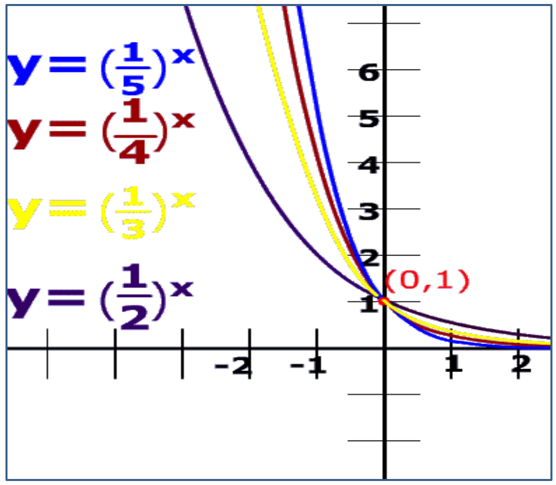
$$a > 1$$

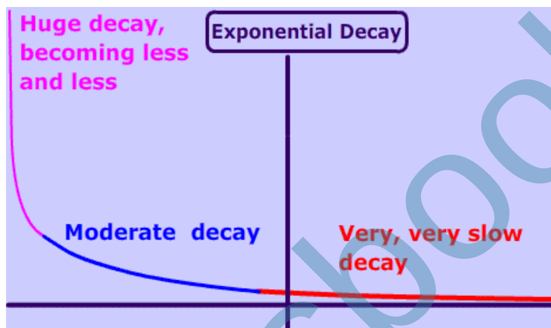




Case of Exponential Decay

$$0 > a > 1$$



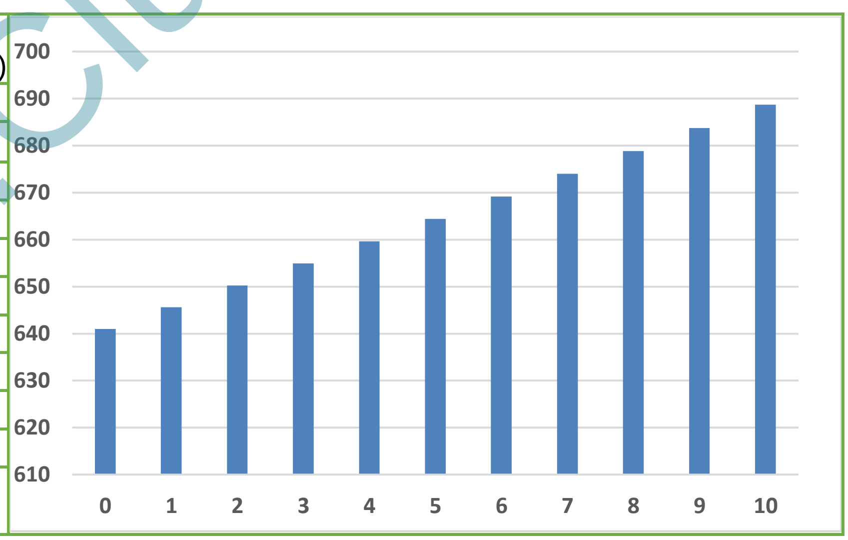


TOPIC 030: POPULATION GROWTH USING GENERAL EXPONENTIAL FUNCTIONS

Europe population function function: $\boldsymbol{P(t) = 641 \cdot (1.0072)^t}$

- $\boldsymbol{P}$ is population in millions,
- $\boldsymbol{t}$ is time in years

Time (t)	Population (P)
0	641
1	645.6152
2	650.2636
3	654.9455
4	659.6611
5	664.4107
6	669.1945
7	674.0127
8	678.8655
9	683.7534
10	688.6764



For $t = 40$, ⇒ year 2000

$P(40) \approx 854$ million

- Whereas, Actual $P(40) = 728$ million.

- Overestimation of 126 million could be due to poor projection.
- Time required for certain level of population (say 900 million).
  $900 = 641 \cdot (1.0072)^t$
  $t = 47.3$ years.
- Detailed steps

$$900 = 641\ (1.0072)^t$$

Solution steps

Use the rules of exponents and logarithms to solve the equation.

$$641 \cdot (1.0072)^t = 900$$

Divide both sides of the equation by 641 .

$$(1.0072)^t = \frac{900}{641}$$

Take the logarithm of both sides of the equation.

$$\log((1.0072)^t) = \log\left(\frac{900}{641}\right)$$

The logarithm of a number raised to a power is the power times the logarithm of the number.

$$t\ \log(1.0072) = \log\left(\frac{900}{641}\right)$$

Divide both sides of the equation by log(1.0072) .

$$t = \frac{\log\left(\frac{900}{641}\right)}{\log(1.0072)}$$

By the change-of-base formula $log(a)/log(b) = log_b(a)$.

$$t = \log_{1.0072}\left(\frac{900}{641}\right)$$

Solution

$$t = \log_{1.0072}\left(\frac{900}{641}\right) \approx 47.3035500975598$$

TOPIC 031: OTHER TYPES: NON-ALGEBRAIC NATURAL EXPONENTIAL FUNCTIONS

Special type of exponential functions

- Represented by using $\boldsymbol{e}$ in the base of the exponent.
- $\boldsymbol{e}$ = exponential
- It has a constant value: **2.718**.
- Also known as 'magical number'.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
4	2.441
12	2.613
365	2.7146
1000	2.7169
10000	2.7184
100000	2.718268
1000000	2.7182804

The Natural Exponential Function:

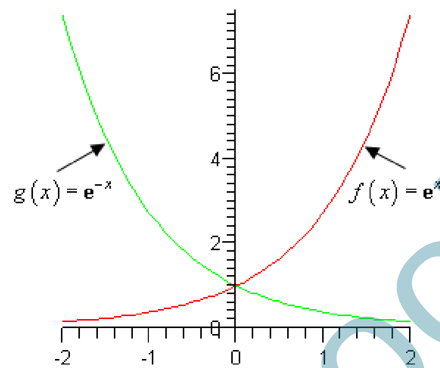
$f(x) = e^x$

a. $e^{-2} \approx 0.1353$

b. $e^{-1} \approx 0.3679$

c. $e^{1} \approx 2.7183$

d. $e^{2} \approx 7.3891$



TOPIC 032: POPULATION GROWTH USING NATURAL EXPONENTIAL FUNCTIONS

Population growth of a region

- $P(t) = P(o).e^{kt}$
- $t = 0 \Rightarrow$ in year 2000 $P(0) = 98{,}632$
- $t = 5 \Rightarrow$ in year 2005 $P(5) = 1{,}09{,}116$
- Substituting:

$$1{,}09{,}116 = (98{,}632)e^{k(5)}$$

- $k = 2.02\%$

Detailed solution

Use the rules of exponents and logarithms to solve the equation.

$$98632\, e^{5k} = 109116$$

Divide both sides of the equation by 98632 .

$$e^{5k} = \frac{27279}{24658}$$

Take the logarithm of both sides of the equation.

$$\log(e^{5k}) = \log\left(\frac{27279}{24658}\right)$$

The logarithm of a number raised to a power is the power times the logarithm of the number.

$$5k\, \log(e) = \log\left(\frac{27279}{24658}\right)$$

Divide both sides of the equation by $\log(e)$.

$$5k = \frac{\log\left(\frac{27279}{24658}\right)}{\log(e)}$$

By the change-of-base formula $log(a)/log(b) = log_b(a)$.

$$5k = \log_e\left(\frac{27279}{24658}\right)$$

Divide both sides of the equation by 5 .

$$k = \frac{\ln\left(\frac{27279}{24658}\right)}{5}$$

Solution

$$k = \frac{\ln\left(\frac{27279}{24658}\right)}{5} \approx 0.0202031567931$$

Forecast for Population Growth after 10 years

- t = 10 ⇒ year 2010.
- $P(t) = (98632)e^{0.02(t)}$
- $P(10) = (98632)e^{0.02(10)}$
- $P(10) \approx 120711$

Time needed for doubling of Population

Let T be that time.

- Double population =

$$= 2 \times (98632)$$
$$P(T) = 197264$$

- $P(T) = (98632)e^{0.02(T)}$
- $197264 = (98632)e^{0.02(T)}$
- $T \approx 34.657$ years

Use the rules of exponents and logarithms to solve the equation.

$$98632\, e^{0.02\,T} = 197264$$

Divide both sides of the equation by 98632 .

$$e^{0.02\,T} = 2$$

Take the logarithm of both sides of the equation.

$$\log(e^{0.02\,T}) = \log(2)$$

The logarithm of a number raised to a power is the power times the logarithm of the number.

$$0.02\,T\,\log(e) = \log(2)$$

Divide both sides of the equation by $\log(e)$.

$$0.02\,T = \frac{\log(2)}{\log(e)}$$

By the change-of-base formula $log(a)/log(b) = log_b(a)$.

$$0.02\,T = \log_e 2$$

Multiply both sides of the equation by 50 .

$$T = \frac{\ln(2)}{0.02}$$

Solution $T = 50\,\ln(2) \approx 34.6573590279973$

Lesson 08

LOGARITHMIC FUNCTIONS AND INVERSE FUNCTIONS

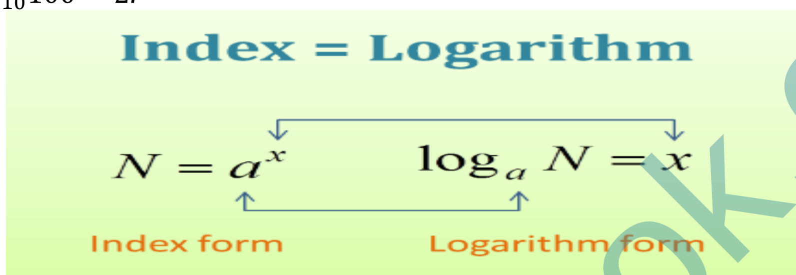
TOPIC 033: OTHER TYPES: NON-ALGEBRAIC LOGARITHMIC FUNCTIONS

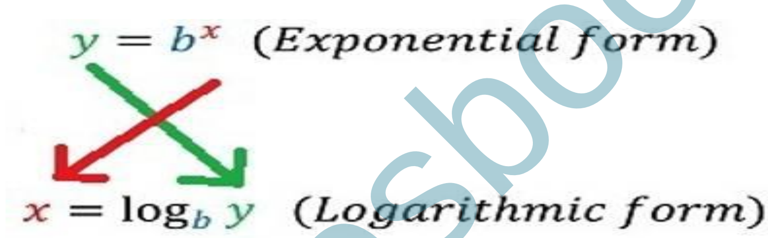
"Exponents in disguise".

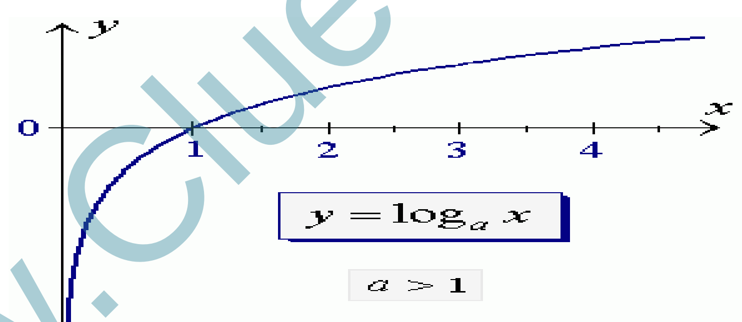
- Output is an exponent.
- If $a^x = N$ is index/exponent form then
- $log_a N = x$ is its logarithmic form.
- Numerically:

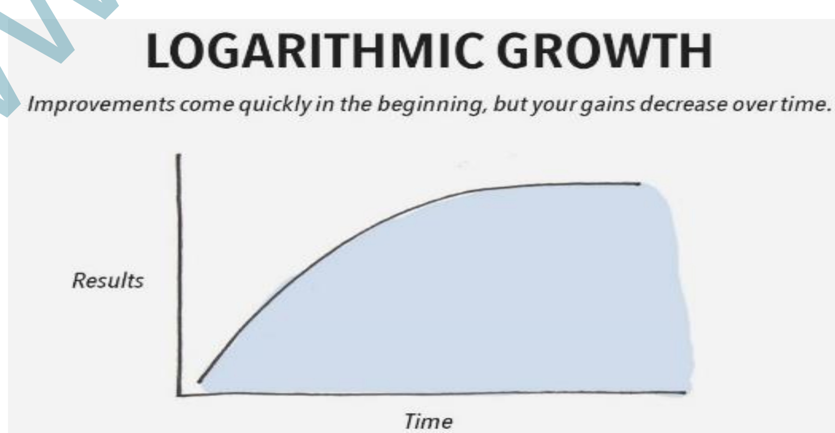
$10^2 = 100$, Where, $a = 10$, $x = 2$ and $N = 100$.

- Then, $log_{10} 100 = 2.$









Law 1 $\log_a(mn) = \log_a m + \log_a n$

Law 2 $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

Law 3 $\log_a(m)^p = p\log_a m$

Some other interesting results

$\log_a\left(\frac{1}{n}\right) = -\log_a n$ $\qquad$ $\log_a(1) = 0$

Nonlinear demand

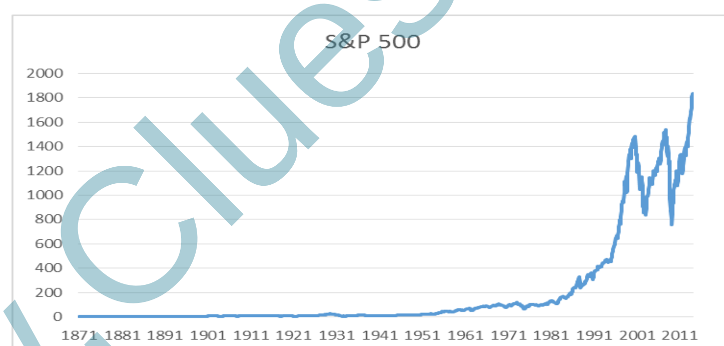
Demand may not be a linear function. A popular nonlinear form takes the form

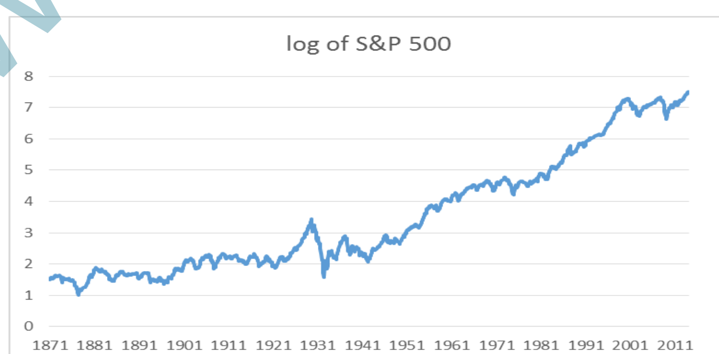
$Q_x = cP_x^{Bx}P_y^{By}M^{BM}H^{BH}$. An example would be

$Q_x = 10P_x^{-1.2}P_y^{3}M^{.5}H^{.3}$.

$\log Q_x = 1 \; - 1.2 \log P_x + 3 \log P_y + .5 \log M + .3 \log H$.

This nonlinear demand is said to be linear in logs.





TOPIC 034: OTHER TYPES: NON-ALGEBRAIC NATURAL LOGARITHMIC FUNCTIONS

"A logarithm to the base e (2.71828 …).".

- Represented using $'\ln'$ instead of $'\log'$.
- Base of natural log is omitted as it is understood.
- Laws of natural logarithm are similar to that of common logarithm.

The laws of natural logarithms

$$\ln a + \ln b = \ln ab$$
$$\ln a - \ln b = \ln \frac{a}{b}$$
$$\ln a^b = b \ln a$$

The Cobb-Douglas production function

- The Cobb-Douglas Production Function:
$$Q_i = B_1 L_i^{B_2} K_i^{B_3}$$
can be transformed into a linear model by taking natural logs of both sides:
$$\ln Q_i = \ln B_1 + B_2 \ln L_i + B_3 \ln K_i$$
- The slope coefficients can be interpreted as elasticities.
  - If $(B_2 + B_3) = 1$, we have constant returns to scale.
  - If $(B_2 + B_3) > 1$, we have increasing returns to scale.
  - If $(B_2 + B_3) < 1$, we have decreasing returns to scale.

TOPIC 035: RATE OF GROWTH OF GNP USING LOGARITHMIC FUNCTIONS

- $GNP_{1990}^{China} = \$1.2 \times 10^{12}$
- Rate of growth of GNP of China (s) = 0.02
- $GNP_{1990}^{USA} = \$5.6 \times 10^{12}$
- Rate of growth of GNP of USA (r) = 0.09
- If the GNP of each country continued to grow exponentially, when would the GNP of the two nations be the same?

Let t denote the number of years after 1990. Assuming continuous exponential growth, when the GNP of the two nations is the same, one must have $1.2 \cdot 10^{12} \cdot e^{0.09t} = 5.6 \cdot 10^{12} \cdot e^{0.02t}$.

$$t = \frac{1}{0.09 - 0.02} \ln \frac{5.6 \cdot 10^{12}}{1.2 \cdot 10^{12}} = \frac{1}{0.07} \ln \frac{14}{3} \approx 22$$

According to this, the two countries would have the same GNP approximately 22 years after 1990

TOPIC 036: INVERSE FUNCTIONS

A reciprocal of a function.

- If original function is:
  $y = f(x)$,
  Then $x = f^{-1}(y)$

Or x = $g\ (y)$

- Economic applications of inverse function include 'Inverse demand function'

$$D = \frac{30}{P^{1/3}}$$

D is a function of P

That is, $D = f(P)$

If we look at the matter from a producer's point of view, however, it may be more natural to treat output as something it can choose and consider the resulting price. The producer wants to know the *inverse* function, in which price depends on the quantity sold.

$$P^{1/3} = 30/D$$

$$(P^{1/3})^3 = (30/D)^3$$

$$P = \frac{27\,000}{D^3}$$

This equation gives us directly the price P corresponding to a given output D. For example, if $D = 10$, then $P = 27\,000/10^3 = 27$. In this case, P is a function $g(D)$ of D, with $g(D) = 27\,000/D^3$.

$$f(P) = 30p^{-1/3} \qquad \text{and} \qquad g(D) = 27\,000D^{-3}$$

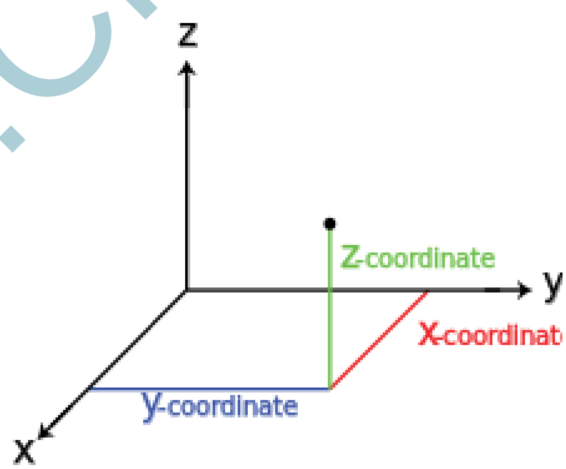
TOPIC 037: FUNCTIONS WITH TWO OR MORE INDEPENDENT VARIABLES

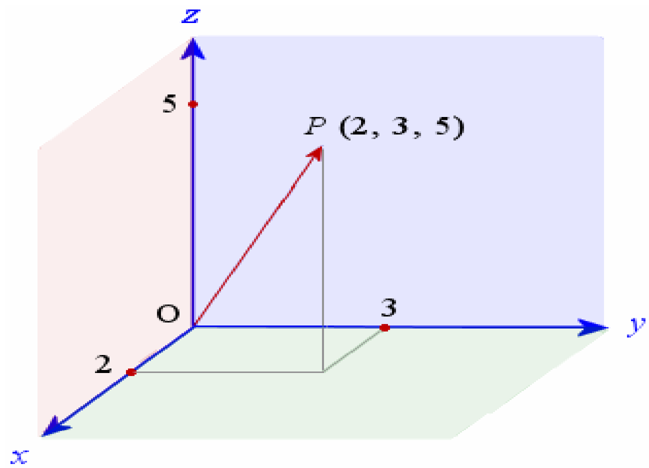
$$z = g(x,\ y)$$

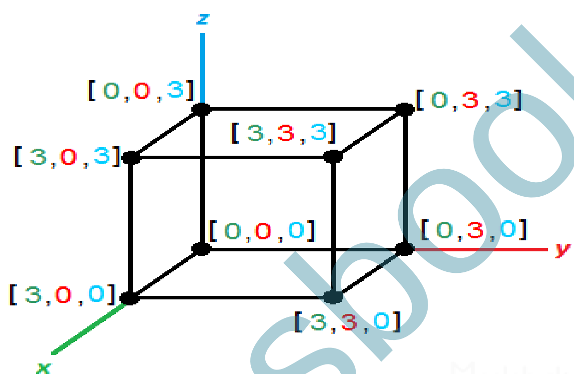
- A given pair of x and y give value of z.
- e.g. Linear specification: z = $ax + by$
- e.g. Quadratic specification:

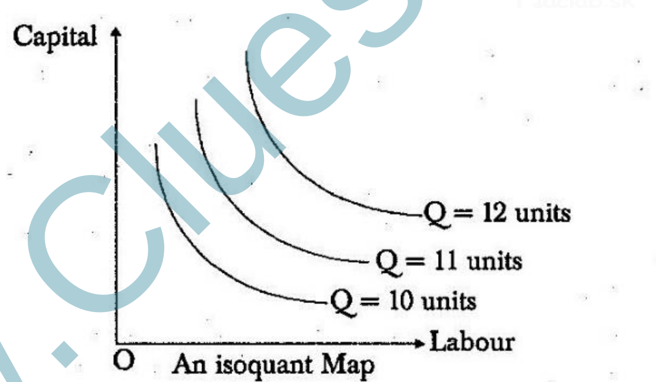
z = $a_o + a_1x + a_2x^2 + b_1x + b_2x^2$

- Ordered triples $(x,\ y, z)$.









- More than two independent variables can also exist.

$$y = g(u, v, w)$$

- e.g. Utility function:

$$U = g(x_1, x_2, x_3)$$

- Ordered quadruples (x_1, x_2, x_3, U).
- Hypersurface – non-graphable.

The demand for sugar in the United States in the period 1929–1935 was estimated by T. W. Schultz, who found that it could be described approximately by the formula

$$x = 108.83 - 6.0294p + 0.164w - 0.4217t$$

Here x, the demand for sugar, is a function of three variables: p (the price of sugar), w (a production index), and t (the date, where $t = 0$ corresponds to 1929).

R. Stone estimated the following formula for the demand for beer in the UK:

$$x = 1.058\, x_1^{0.136} x_2^{-0.727} x_3^{0.914} x_4^{0.816}$$

Here the quantity demanded x is a function of four variables: x_1 (the income of the individual), x_2 (the price of beer), x_3 (a general price index for all other commodities), and x_4 (the strength of the beer).

- Generally speaking, $'n'$ number of independent variables.

$$y = g(x_1, x_2, x_3, \ldots, x_n)$$

- e.g. Utility function:

$U = U(x_1, x_2, x_3, \ldots, x_n)$.

TOPIC 038: SURFACES AND DISTANCE IN GRAPHS OF TWO OR MORE INDEPENDENT VARIABLES

Surfaces

- $f(x, y) = c$ makes a point in graph.
- $g(x, y, z) = c$ makes a surface in graph.

(the general equation for a plane in space)

$$ax + by + cz = d$$

(with a, b, and c not all 0)

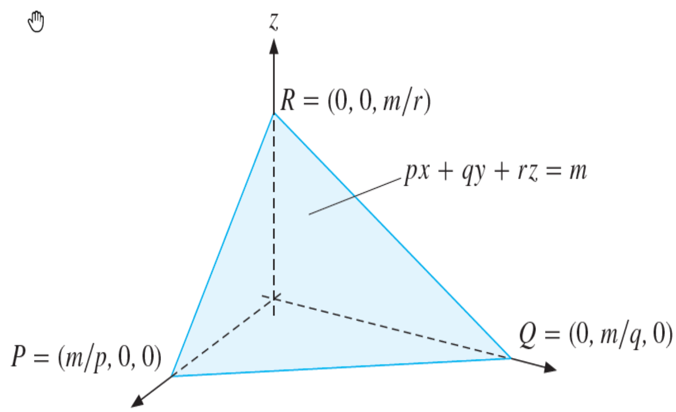
Let us rename the coefficients and consider the equation

$$px + qy + rz = m$$

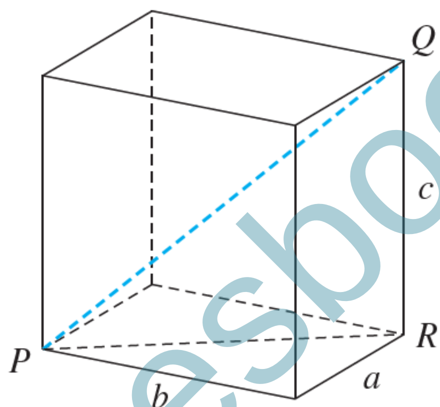
where p, q, r, m are all positive. This equation can be given an economic interpretation. Suppose a household has a total budget of m to spend on three commodities, whose prices are respectively p, q, and r per unit.

in most cases one also has $x \geq 0$, $y \geq 0$, and $z \geq 0$

$$B = \{(x, y, z) : px + qy + rz \leq m,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$$



The Distance Formula



Consider a rectangular box with edges of length a, b, and c
Pythagoras's
theorem, $(PR)^2 = a^2 + b^2$

Lesson 09

EQUATIONS AND TYPES OF EQUATIONS

TOPIC 039: EQUATIONS AND IDENTITIES

Equations

Mathematical expression with equality. e.g.

$$y = mx + c$$
$$y = ax^2 + bx + c$$

- True for certain values of x.
- $2x + 4 = x + 2$ is true only for $x = -2$.
- Not for other values.

Economic Examples of Equations

- Demand function

$$D = \alpha - \beta . P$$

- Production function

$$Q = A . L^{\alpha} . K^{\beta}$$

- Cost function

$$C = w . L + r . K$$

- Optimization condition

$$MC = MR$$

- Market equilibrium condition

$$Q_d = Q_s$$

Identities

Etymology: Latin: *Idem* (same).

- Mathematical equalities that are true for all values of x.
- e.g.

$$1 \equiv \sin^2 \theta + \cos^2 \theta$$

- $\equiv$ is used to represent an identity.

$$\{\sin^2 30^o + \cos^2 30^o = \sin^2 60^o + \cos^2 60^o = \sin^2 90^o + \cos^2 90^o = 1\}$$

Economic Example of Identity

Equation of profit:

$$Profit = Revenue - Cost$$
$$\pi = R - C$$

- Deducting cost from revenue will always give profit, therefore:
- $\pi \equiv R - C$

TOPIC 040: TYPES OF EQUATIONS IN ECONOMICS: DEFINITIONAL EQUATIONS

Reflection of a definition in an equation.

- An Identity having exactly same meaning.

Economic Examples

Revenue Function:

$$\pi \equiv R - C$$

- Is read as, "Revenue is identically equal to the difference of revenue and cost".
- However, (=) is also used.

$$\pi = R - C$$

- Here $R = P . Q$

$$\pi \equiv R - C$$

- Regardless of the positivity/negatively of answer.

$$\pi \equiv R - C \lesseqgtr 0$$

- Identity of profit function remains intact even if answer is negative (loss) or zero (breakeven).

National Income Accounting Identity

$$Y \equiv C + I + G + (X - M)$$

- Is read as: "Sum of expenditures by consumers, investors & government, and net exports is equation to national income", under expenditure approach.

<u>TOPIC 041: FISCAL SURPLUS AND FISCAL DEFICIT USING EQUATIONS</u>

Fiscal Deficit/Surplus

Fiscal Deficit

$$\boldsymbol{Fiscal\ Deficit \equiv Government\ Revenue - GovernmentExpenditures < 0}$$

- Is read as: "Fiscal deficit is identically equal to the difference of government revenue and government expenditure", while former is smaller than latter.

$$\boldsymbol{FD \equiv GR - GE < 0}$$

- e.g. $\boldsymbol{FD \equiv 200M - 250M \equiv -50M < 0}$
- Government is suffering from 50M of fiscal deficit.

Fiscal Surplus

$$\boldsymbol{Fiscal\ Surplus \equiv Government\ Revenue - GovernmentExpenditures > 0}$$

- Is read as: "Fiscal Surplus is identically equal to the difference of government revenue and government expenditure", while former is greater than latter.

$$\boldsymbol{FD \equiv GR - GE > 0}$$

- e.g. $\boldsymbol{FD \equiv 250M - 200M \equiv 50M < 0}$
- Government is 50M of fiscal surplus.

Neither Fiscal Deficit nor Surplus

$$\boldsymbol{\equiv Government\ Revenue - GovernmentExpenditures = 0}$$

- Is read as: "Neither fiscal deficit nor surplus exist when there is no difference b/w government revenue and government expenditure".

$$\boldsymbol{FD \equiv GR - GE = 0}$$

- e.g. $\boldsymbol{FD \equiv 250M - 250M \equiv 0M = 0}$
- Government is neither having any fiscal deficit nor fiscal surplus.

- Fiscal deficit/surplus using tax revenue function.

$$\boldsymbol{T = T_o + tY}$$
$$T = 240 + 0.2(Y)$$

If $Y = 850$

$$T = 240 + 0.2(850)$$
$$T = 410$$

If $G_o = 330$,

$$= T - G_o$$

$= 410 - 330 = 80$

Budget surplus = 80.

- If G by 60, T due to in government spending multiplier ($\Delta Y = 150$).
- New G= Old $\text{G} + \Delta G = 330 + 60 = 390$.
- $\Delta T = 0.2(150) = 30$
- New Y= Old $\text{Y} + \Delta Y = 850 + 150 = 1000$.
- New T= $240 + 0.2(1000) = 440$.
- New budget surplus= $440 - 390 = 50$ (compared with 80).

TOPIC 042: TYPES OF EQUATIONS IN ECONOMICS: BEHAVIORAL EQUATIONS

Specifies the manner in which dependent variables behaves in response to changes in independent variable(s).

- Can include technological and legal aspects.
- Such behavior can be either human or non-human.

- Human behavior: Aggregate consumption in relation to national income.

$$C = C_o + MPC.Y$$

- Non-human behavior: Total Cost in relation to output.

$$TC = FC + VC$$
$$TC = a + b.Q$$

- Consider two cost functions:

$$C = 75 + 10Q$$
$$C = 110 + Q^2$$

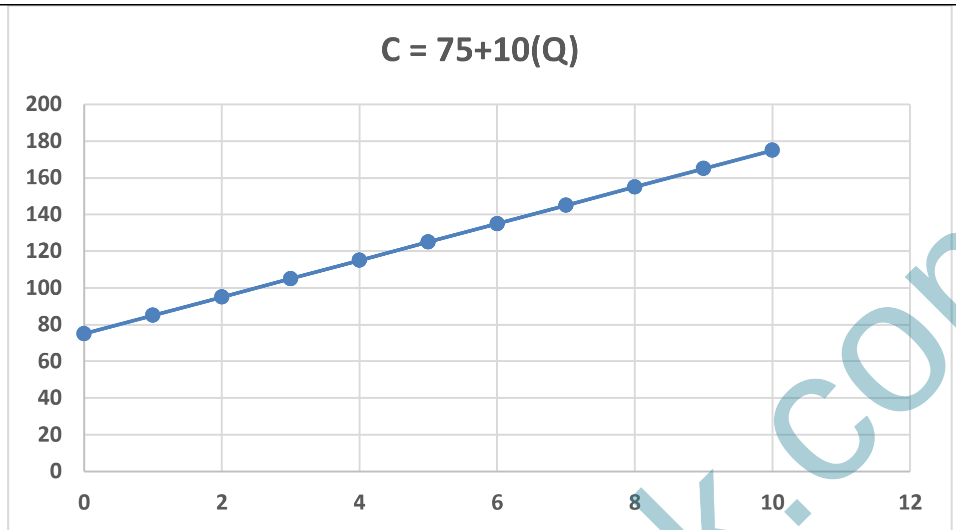
- Fixed cost(FC) = $(C)_{Q=0}$
- FC are 75 and 110, respectively.
- First function has linear relationship.
- While second has quadratic.

- Consider two cost functions:

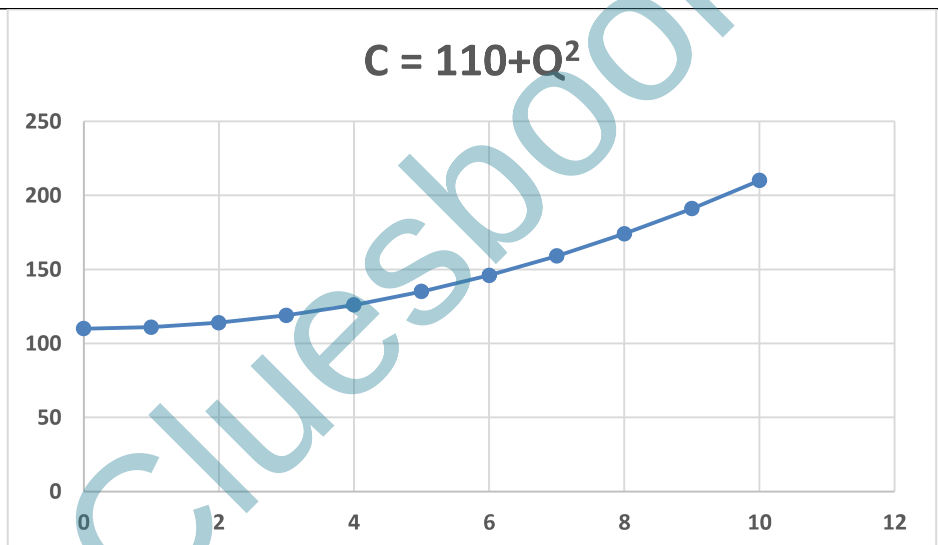
$$C = 75 + 10Q$$
$$C = 110 + Q^2$$

- Fixed cost(FC) = $(C)_{Q=0}$
- FC are 75 and 110, respectively.
- First function has linear relationship.
- While second has quadratic.

C=75+10Q	
Q	C
0	75
1	85
2	95
3	105
4	115
5	125
6	135
7	145
8	155
9	165
10	175



$C=110+Q^2$	
Q	C
0	110
1	111
2	114
3	119
4	126
5	135
6	146
7	159
8	174
9	191
10	210



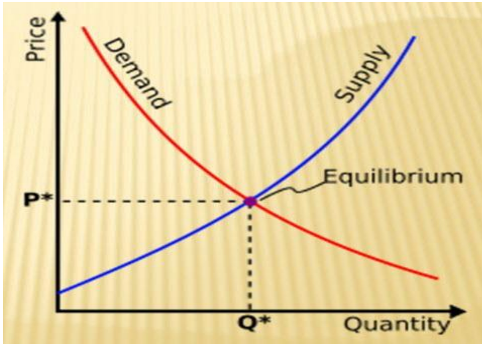
TOPIC 043: TYPES OF EQUATIONS IN ECONOMICS: CONDITIONAL EQUATIONS

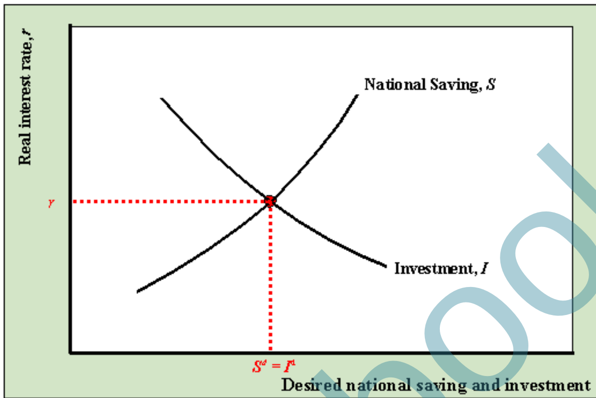
Specifies a requirement to be satisfied.

- To specify equilibrium, an equilibrium condition should be specified.
- Two famous equilibrium conditions are:

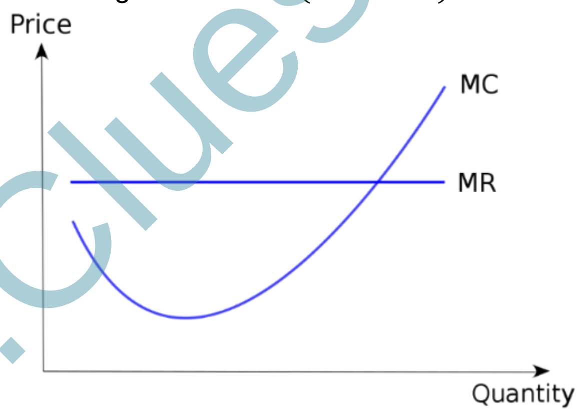
Quantity demanded = Quantity supplied ($Q_d = Q_s$).

Desired savings = Desired investment ($S = I$).





- Optimization conditions is also an example:
    - Marginal cost = Marginal revenue ($MC = MR$).



TOPIC 044: STRUCTURAL AND REDUCED FORM EQUATIONS

Assume following equations:

$Y = C + \bar{I}$ ....(1)

$C = a + bY$ ....(2)

Y = National income

C = Consumption

$\bar{I}$ = Autonomous investment

a and b are parameters.

Equation (1) & (2) are structural form equations.

- Putting value of C in eq. (1).

$$Y = a + bY + \bar{I}$$
$$Y - bY = a + \bar{I}$$
$$Y(1-b) = a + \bar{I}$$
$$Y^* = \frac{a + \bar{I}}{(1-b)}$$
$$\boldsymbol{Y^* = \frac{a}{(1-b)} + \left\{\frac{1}{(1-b)}\right\}\bar{I}}$$

- Value of Y (endogenous variable) in terms of exogenous variable ($\bar{I}$) and parameters (a & b).
- Reduced form equation

- Value of other endogenous variable (C):

$C = a + bY$

$$C^* = a + bY^*$$
$$Y^* = \frac{a}{(1-b)} + \left\{\frac{1}{(1-b)}\right\}\bar{I}$$
$$C^* = a + b\left[\frac{a}{(1-b)} + \left\{\frac{1}{(1-b)}\right\}\bar{I}\right]$$
$$= a + \left\{\frac{ab}{(1-b)}\right\} + \left\{\frac{b}{(1-b)}\right\}\bar{I}$$
$$= \left\{\frac{a(1-b) + ab + b\bar{I}}{(1-b)}\right\}$$

$\boldsymbol{C^*} = \left\{\frac{a-ab+ab+b\bar{I}}{(1-b)}\right\} = \left\{\boldsymbol{\frac{a+b\bar{I}}{(1-b)}}\right\}$

Numerical results:

- If $\bar{I} = 100$, $a = 500$ and $b = 0.8$, then

$$Y^* = \frac{a}{(1-b)} + \left\{\frac{1}{(1-b)}\right\}\bar{I}$$
$$Y^* = \frac{500}{(1-0.8)} + \left\{\frac{1}{(1-0.8)}\right\}(100)$$

$\boldsymbol{Y^*} = \mathbf{3000}$

$$C^* = \left\{\frac{a + b\bar{I}}{(1-b)}\right\}$$
$$C^* = \left\{\frac{500 + (0.8)(100)}{(1-0.8)}\right\}$$
$$\boldsymbol{C^* = 2900}$$

Lesson 10

PARTIAL LINEAR MARKET EQUILIBRIUM

TOPIC 045: CONSTRUCTING A PARTIAL LINEAR MARKET EQUILIBRIUM

- Three variables; $\boldsymbol{Q_d}$, $\boldsymbol{Q_s}$ and $\boldsymbol{P}$:
- Specify conditional equation for equilibrium:

$\boldsymbol{Q_d} = \boldsymbol{Q_s}$ ……..(1)
Or

$$\boldsymbol{Q_d - Q_s = 0}$$

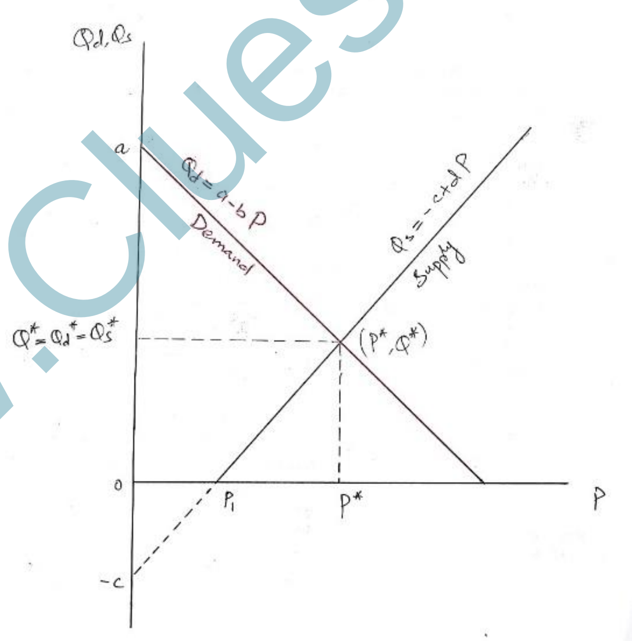
- Excess demand is equal to zero.

- Behavioral equations of $\boldsymbol{Q_d}$ and $\boldsymbol{Q_s}$.

$\boldsymbol{Q_d} = a - b.P$ ……………(2)
$\boldsymbol{Q_s} = -c + d.P$………….(3)
Where $(a,\ b,\ c,\ d) > 0$

- Where, $\boldsymbol{a}$ is the intercept of $\boldsymbol{Q_d}$. $\boldsymbol{b}$ is the slope of $\boldsymbol{Q_d}$ which is negative.
- $\boldsymbol{c}$ is the intercept of $\boldsymbol{Q_s}$. $\boldsymbol{d}$ is the slope of $\boldsymbol{Q_d}$ which is positive.

- $\boldsymbol{Q_s}$ has horizontal intercept at $\boldsymbol{P_1}$ which is reservation price – lowest **price** at which a seller is willing to sell.
- Contrary to convention where $\boldsymbol{P}$ is plotted on y-axis and $\boldsymbol{Q_s}$, $\boldsymbol{Q_d}$ on x-axis.
- Convention is based on inverse demand function.
- Mathematically justified.



TOPIC 046: SOLVING PLMM USING ELIMINATION OF VARIABLE METHOD

Equate behavioral equations of $\boldsymbol{Q_d}$ and $\boldsymbol{Q_s}$ by assuming
$\boldsymbol{Q_d} = \boldsymbol{Q_s} = \boldsymbol{Q}$.
$\boldsymbol{Q} = a - b.P$ ……………(2')
$\boldsymbol{Q} = -c + d.P$………….(3')

$$a - b.P = -c + d.P$$
$$a + c = b.P + d.P$$
$$a + c = (b + d).P$$
$$\frac{a + c}{(b + d)} = P$$

Equilibrium price

$\boldsymbol{P^*} = \frac{\boldsymbol{a+c}}{\boldsymbol{b+d}}$; $(b + d > 0)$

- Substitute P^* in $\boldsymbol{Q} = a - b.P$ (or $\boldsymbol{Q} = -c + d.P$).

$$\boldsymbol{Q} = a - b.\left(\frac{a + c}{b + d}\right)$$
$$= a - \frac{b(a + c)}{b + d}$$
$$= \frac{a(b + d) - b(a + c)}{b + d}$$
$$= \frac{ab + ad - ab - bc}{b + d}$$

$\boldsymbol{Q^*} = \frac{\boldsymbol{ad-bc}}{\boldsymbol{b+d}}$, $(ad - bc > 0)$

OR $(ad > bc)$ for $\boldsymbol{Q^*} > \boldsymbol{0}$

- **In set notation**

$$\boldsymbol{D} = \{(\boldsymbol{P}, \boldsymbol{Q}) | \boldsymbol{Q} = \boldsymbol{a} - \boldsymbol{b}.\boldsymbol{P}\}$$
$$\boldsymbol{S} = \{(\boldsymbol{P}, \boldsymbol{Q}) | \boldsymbol{Q} = -\boldsymbol{c} + \boldsymbol{d}.\boldsymbol{P}\}$$
$$\boldsymbol{D} \cap \boldsymbol{S} = (\boldsymbol{P^*}, \boldsymbol{Q^*})$$

TOPIC 047: SHIFTS IN DEMAND IN MARKET EQUILIBRIUM

Assume demand & supply functions:

$\boldsymbol{D = 100 - P}$ &

$\boldsymbol{S = 10 + 2P}$

$$\boldsymbol{D = S}$$
$$\boldsymbol{100 - P = 10 + 2P}$$
$$\boldsymbol{90 = 3P}$$
$$\underline{\boldsymbol{P^* = 30}}$$
$$\boldsymbol{Q^* = D = 100 - 30}$$
$$\underline{\boldsymbol{Q^* = 70}}$$

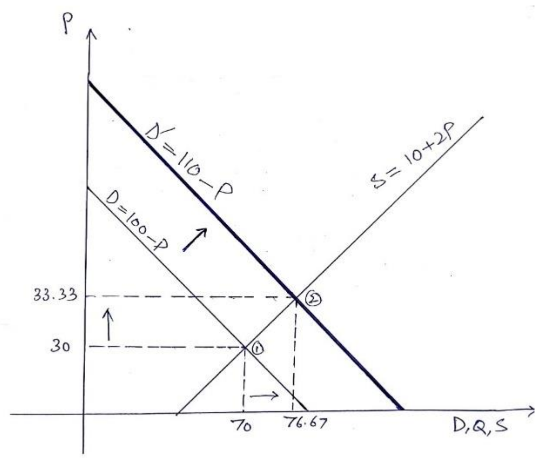
- Equilibrium price and equilibrium output are 30 and 70, respectively.

- Any exogenous factor (e.g. increased income) increases the demand shifting its curve to the right side:

$\boldsymbol{D' = 110 - P}$

$$\boldsymbol{D = S}$$
$$\boldsymbol{110 - P = 10 + 2P}$$
$$\boldsymbol{100 = 3P}$$
$$\underline{\boldsymbol{P^* = 33.3}}$$
$$\boldsymbol{Q^* = D = 110 - 33.3}$$
$$\underline{\boldsymbol{Q^* = 76.7}}$$

- New equilibrium price and equilibrium output are 33.3 and 76.7, respectively.



TOPIC 048: SHIFTS IN SUPPLY IN MARKET EQUILIBRIUM

Assume demand & supply functions:

$\boldsymbol{D = 100 - P}$ &

$\boldsymbol{S = 10 + 2P}$

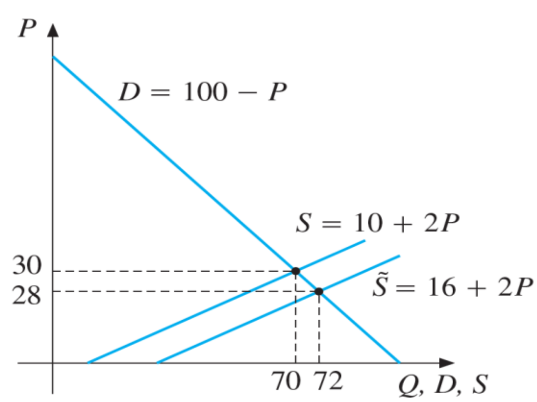
$$\boldsymbol{D = S}$$
$$\boldsymbol{100 - P = 10 + 2P}$$
$$\boldsymbol{90 = 3P}$$
$$\underline{\boldsymbol{P^* = 30}}$$
$$\boldsymbol{Q^* = D = 100 - 30}$$
$$\underline{\boldsymbol{Q^* = 70}}$$

- Equilibrium price and equilibrium output are 30 and 70, respectively.
- Any exogenous factor (e.g. improved technology) increases the supply shifting its curve to the right side:

$\boldsymbol{\tilde{S} = 16 + 2P}$

$$\boldsymbol{D = \tilde{S}}$$
$$\boldsymbol{100 - P = 16 + 2P}$$
$$\boldsymbol{84 = 3P}$$
$$\underline{\boldsymbol{P^* = 28}}$$
$$\boldsymbol{Q^* = \tilde{S} = 16 + 2P}$$
$$\underline{\boldsymbol{Q^* = 72}}$$

- Equilibrium price and equilibrium output are 28 and 72, respectively.



TOPIC 049: EFFECT OF TAX ON PRODUCER ON PARTIAL MARKET EQUILIBRIUM

Assume demand & supply functions:

$D = 150 - \frac{1}{2}P$ &

$S = 20 + 2P$

$$D = S$$
$$150 - \frac{1}{2}P = 20 + 2P$$
$$130 = 5P$$
$$\underline{P^* = 52}$$
$$Q^* = D = 150 - \frac{1}{2}(52)$$
$$\underline{Q^* = 124}$$

- Rs. 2 per unit tax on producer:
- Let the tax be 't'

$S^t = 20 + 2(P - t)$

$S^t = 20 + 2(P - 2)$

$$S^t = 16 + 2P$$
$$D = S^t$$
$$150 - \frac{1}{2}P = 16 + 2P$$
$$\underline{P^* = 53.6}$$
$$Q^* = S^t = 16 + 2(53.6)$$
$$\underline{Q^* = 123.2}$$

- Revenue of producer before tax (R^{bt}) and after tax (R^{at}):

$$R^{bt} = P^{bt} \times Q^{bt}$$
$$= 52 \times 124$$
$$\underline{R^{bt} = 6448}$$
$$R^{at} = P^{at} \times Q^{at}$$
$$P^{at} = P - t = 51.6$$
$$= 51.6 \times 123.2$$
$$\underline{R^{at} = 6357.12}$$

- Change in producer revenue = $\underline{-90.88}$

TOPIC 050: EFFECT OF TAX ON CONSUMER ON PARTIAL MARKET EQUILIBRIUM

Assume demand & supply functions:

$D = 150 - \frac{1}{2}P$ &

$S = 20 + 2P$

$$D = S$$
$$150 - \frac{1}{2}P = 20 + 2P$$
$$130 = 5P$$
$$\underline{P^* = 52}$$
$$Q^* = D = 150 - \frac{1}{2}(52)$$
$$\underline{Q^* = 124}$$

- Rs. 2 per unit tax on consumer:

$D^t = 150 - \frac{1}{2}(P + 2)$

$$D^t = 149 - \frac{1}{2}(P)$$
$$D^t = S$$
$$149 - \frac{1}{2}(P) = 20 + 2P$$
$$P^* = 51.6$$
$$Q^* = D^t = 149 - \frac{1}{2}(51.6)$$
$$Q^* = 123.2$$

Lesson 11

GENERAL EQUILIBRIUM AND NATIONAL INCOME EQUILIBRIUM

TOPIC 051: PARTIAL MARKET EQUILIBRIUM-A NON-LINEAR MODEL

Assume demand & supply functions:

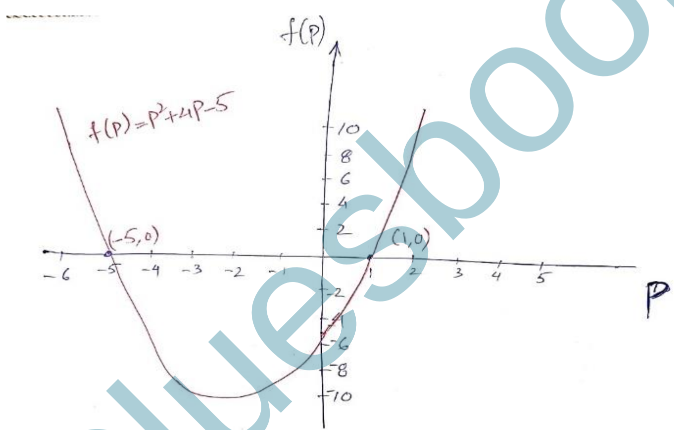
$\boldsymbol{Q_d = 4 - P^2}$ &

$$Q_s = 4P - 1$$
$$Q_d = Q_s$$
$$4 - P^2 = 4P - 1$$
$$P^2 + 4P - 5 = 0$$
$$P^2 + 5P - P - 5 = 0$$
$$P(P+5) - (P+5) = 0$$
$$(P+5)\,(P-1) = 0$$

Either $(P + 5) = 0$

Or $(P - 1) = 0$

$$P = 1, \; -5$$
$$P = 1$$



- $ax^2 + bx + c = 0$ is the standard form of quadratic (non-linear) function.
- It can also be solved using quadratic formula:
-

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$P^2 + 4P - 5 = 0$$

- Here, $x = P$, $a = 1$, $b = 4$ and $c = -5$.

$$P = \frac{-4 \pm \sqrt{4^2 - 4(1)(-5)}}{2(1)}$$

$P = \frac{-4 \pm \sqrt{4^2 - 4(1)(-5)}}{2(1)}$

$P = \frac{-4 \pm \sqrt{16+20}}{2}$

$P = \frac{-4 \pm \sqrt{36}}{2} = \frac{-4 \pm 6}{2}$

$P = \frac{-4+6}{2}\;,\;\frac{-4-6}{2}$

$P = 1, -5$

- Other nonlinear models may be cubic, quartile etc.

$ax^3 + bx^2 + cx + d = 0$
$ax^4 + bx^3 + cx^2 + dx + e = 0$

TOPIC 052: GENERAL MARKET EQUILIBRIUM: GENERAL FORM OF TWO GOOD CASE

Assume demand & supply functions:

Good – 1

$Q_{d1} = a_0 + a_1P_1 + a_2P_2$

$Q_{s1} = b_0 + b_1P_1 + b_2P_2$

$Q_{d1} = Q_{s1}$ OR
$Q_{d1} - Q_{s1} = 0$ (No excess demand of Good 1)

Good – 2

$Q_{d2} = \alpha_0 + \alpha_1P_1 + \alpha_2P_2$

$Q_{s2} = \beta_0 + \beta_1P_1 + \beta_2P_2$

$Q_{d2} = Q_{s2}$ OR
$Q_{d2} - Q_{s2} = 0$ (No excess demand of Good 2)

Good 1 $\quad Q_{d_1} = Q_{s_1}$

$$a_0 + a_1P_1 + a_2P_2 = b_0 + b_1P_1 + b_2P_2$$

Rearranging.

$$(a_0 - b_0) + (a_1 - b_1)P_1 + (a_2 - b_2)P_2 = 0 \quad \text{—} ①$$

Good 2 $\quad Q_{d_2} = Q_{s_2}$

$$\alpha_0 + \alpha_1P_1 + \alpha_2P_2 = \beta_0 + \beta_1P_1 + \beta_2P_2$$

$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)P_1 + (\alpha_2 - \beta_2)P_2 = 0 \quad \text{—} ②$$

To Simplify, Let $a_i - b_i = c_i$ & $\alpha_i - \beta_i = \gamma_i$

Therefore Equation ① & ② become.

$$c_0 + c_1P_1 + c_2P_2 = 0 \quad \text{—} ①'$$

$$\gamma_0 + \gamma_1P_1 + \gamma_2P_2 = 0 \quad \text{—} ②'$$

Rearranging:

$$c_1 P_1 + c_2 P_2 = -c_0 \quad \text{——} (1)''$$

$$\gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0 \quad \text{——} (2)''$$

Using Equation (1)''

$$P_2 = \frac{-c_0 - c_1 P_1}{c_2} \quad \text{——} (3)$$

Putting value of P_2 in eq (2)''.

$$\gamma_1 P_1 + \gamma_2 \left(\frac{-c_0 - c_1 P_1}{c_2} \right) = -\gamma_0$$

$$\frac{c_2 \gamma_1 P_1 + \gamma_2 (-c_0 - c_1 P_1)}{c_2} = -\gamma_0$$

$$c_2 \gamma_1 P_1 - c_0 \gamma_2 - c_1 \gamma_2 P_1 = -c_2 \gamma_0$$

$$c_2 \gamma_0 - c_0 \gamma_2 = c_1 \gamma_2 P_1 - c_2 \gamma_1 P_1$$

- The equilibrium outputs of Good -1 (Q_1^*)and Good -2 (Q_2^*) can be found by putting equilibrium values of their prices i.e P_1^* and P_2^*.

TOPIC 053: GENERAL MARKET EQUILIBRIUM: NUMERICAL SOLUTION OF TWO GOOD CASE

Assume demand & supply functions:

Good – 1

$Q_{d1} = 10 - 2P_1 + P_2$

$$Q_{s1} = -2 + 3P_1$$

$$Q_{d1} = Q_{s1}$$

Good – 2

$Q_{d2} = 15 + P_1 - P_2$

$$Q_{s1} = -1 + 2P_2$$

$$Q_{d2} = Q_{s2}$$

- Solving for Good – 1:

$$10 - 2P_1 + P_2 = -2 + 3P_1$$

$10 + 2 - 2P_1 - 3P_1 + P_2 = 0$

$$\underline{12 - 5P_1 + P_2 = 0}$$

- Solving for Good – 2:

$$15 + P_1 - P_2 = -1 + 2P_2$$

$15 + 1 + P_1 - 2P_2 - P_2 = 0$

$\underline{16 + P_1 - 3P_2 = 0}$

Solving underlined equations, simultaneously:

$\underline{12 - 5P_1 + P_2 = 0}$

$\underline{16 + P_1 - 3P_2 = 0}$

$P_1^* = {}^{52}/_{14}$ & $P_2^* = {}^{92}/_{14}$

Put P_1^* in Q_{S1} for Q_1^*:

$Q_{S1} = -2 + 3P_1$

$Q_{S1} = -2 + 3\left({}^{52}/_{14}\right)$

$Q_1^* = Q_{S1} = {}^{64}/_{7}$

Put P_2^* in Q_{S2} for Q_2^*:

$$Q_{S2} = -1 + 2P_2$$

$Q_2^* = Q_{S2} = -1 + 2({}^{92}/_{14}) = {}^{85}/_{7}$

$\{(P_1^*, Q_1^*), (P_2^*, Q_2^*)\} = \{({}^{52}/_{14}, {}^{64}/_{7}), ({}^{92}/_{14}, {}^{85}/_{7})\}$

TOPIC 054: GENERAL MARKET EQUILIBRIUM: N-GOOD CASE

Transition from partial equilibrium analysis to general-equilibrium analysis.

- Two goods case can be generalized to n-number of goods.
- $Q_{di} = Q_{di}(P_1, P_2, \ldots, P_n)$

& $Q_{si} = Q_{si}(P_1, P_2, \ldots, P_n)$

Where, $(i = 1, 2, 3, \ldots, n)$.

- Excess demands $(Q_{di} - Q_{si})$ for n- goods (complete market).

- $Q_{di}(P_1, P_2, \ldots, P_n) - Q_{si}(P_1, P_2, \ldots, P_n) = 0$
- Let $E_i = (Q_{di} - Q_{si})$
- Then $E_i(P_1, P_2, \ldots, P_n) = 0$
- Solved simultaneously, these n equations can determine the n equilibrium prices P^*.

TOPIC 055: NATIONAL INCOME EQUILIBRIUM

Application of algebra to macroeconomic analysis.

- Simple Keynesian model.

$$Y = C + I_0 + G_0$$
$$C = a + bY$$
$$(a > 0, \;\; 0 < b < 1)$$

- Endogenous variables: Y and C represent national Income & consumption respectively.
- Exogenous variables: I_0 and G_0 represent autonomous investment & government expenditure respectively.
- Second equation is a behavioral equation showing the behavior of consumption with respect to income.

- Substituting consumption function in national income equation.

$$Y = C + I_0 + G_0$$
$$C = a + bY$$
$$Y = (a + bY) + I_0 + G_0$$
$$Y - bY = a + I_0 + G_0$$
$$Y(1 - b) = a + I_0 + G_0$$
$$Y^* = \frac{a + I_0 + G_0}{(1 - b)}$$

- Solution is in terms of exogenous variables and parameters.
  $b \neq 1$, for Y^* to be defined.
- Equilibrium level of the other endogenous variable (consumption).
- Substitute Y^* in consumption function.

$$C = a + bY$$
$$C = a + b\left(\frac{a + I_0 + G_0}{(1 - b)}\right)$$

$$= \frac{a(1-b) + b(a + I_0 + G_0)}{(1-b)}$$
$$= \frac{a - ab + ab + bI_0 + bG_0}{(1-b)}$$
$$C^* = \frac{a + b(I_0 + G_0)}{(1-b)}$$

- $b \neq 1$, for Y^* to be defined.

TOPIC 056: NATIONAL INCOME EQUILIBRIUM WITH INDUCED AND AUTONOMOUS TAX

- Macroeconomic analysis with autonomous tax (d) & induced tax ($t.Y$).
- Keynesian model.

$$Y = C + I_0 + G_0$$
$$C = a + b(Y - T)$$
$$T = d + t(Y)$$
$$(a > 0,\ \ 0 < b < 1)$$
$$(d > 0,\ \ 0 < t < 1)$$

- Endogenous variables: Y, C and T represent national income, consumption & taxes respectively.
- Exogenous variables: I_0 and G_0 represent autonomous investment & government expenditure respectively.
- 2nd and 3rd equations are behavioral equations showing the behavior of consumption and taxes with respect to income.

- Substituting consumption function and tax function in national income equation.

$$Y = C + I_0 + G_0$$
$$Y = \{a + b(Y - T)\} + I_0 + G_0$$
$$Y = [a + b\{Y - (d + tY)\}] + I_0 + G_0$$
$$Y = a + bY - bd - btY + I_0 + G_0$$
$$Y - bY + btY = a - bd + I_0 + G_0$$
$$Y(1 - b + bt) = a - bd + I_0 + G_0$$
$$Y = \frac{a - bd + I_0 + G_0}{1 - b + bt}$$
$$Y^* = \frac{a - bd + I_0 + G_0}{1 - b(1-t)}$$

- Equilibrium level of taxes.

$$T = d + tY^*$$
$$T = d + t\left\{\frac{a - bd + I_0 + G_0}{1 - b(1-t)}\right\}$$
$$T = \frac{d\{1 - b(1-t)\} + t(a - bd + I_0 + G_0)}{1 - b(1-t)}$$
$$T = \frac{d - bd + bdt + at - bdt + I_0 t + G_0 t)}{1 - b(1-t)}$$
$$T^* = \frac{d(1-d) + t(a + I_0 + G_0)}{1 - b(1-t)}$$

- Equilibrium level of consumption.

$$C^* = Y^* - I_0 - G_0$$

$$C^* = \left\{\frac{a - bd + I_0 + G_0}{1 - b(1-t)}\right\} - I_0 - G_0$$
$$C^* = \frac{(a - bd + I_0 + G_0) - I_0\{1 - b(1-t)\} - G_0\{1 - b(1-t)\}}{1 - b(1-t)}$$
$$C^* = \frac{a - bd + I_0 + G_0 - I_0 + bI_0(1-t) - G_0 + bG_0(1-t)}{1 - b(1-t)}$$
$$C^* = \frac{a - bd + I_0 + G_0 - I_0 + bI_0 - btI_0 - G_0 + bG_0 - btG_0}{1 - b(1-t)}$$
$$C^* = \frac{a - bd + bI_0 - btI_0 + bG_0 - btG_0}{1 - b(1-t)}$$
$$C^* = \frac{a - bd + b(I_0 - tI_0 + G_0 - tG_0)}{1 - b(1-t)}$$
$$C^* = \frac{a - bd + b\{I_0(1-t) + G_0(1-t)\}}{1 - b(1-t)}$$
$$C^* = \frac{a - bd + b(1-t)(I_0 + G_0)}{1 - b(1-t)}$$

TOPIC 057: NATIONAL INCOME EQUILIBRIUM WITH PROPORTION OF GOVERNMENT EXPENDITURE

Macroeconomic analysis with proportion of government expenditure.

- Keynesian model.

$$Y = C + I_0 + G$$
$$C = a + b(Y - T_0)$$
$$G = gY$$
$$(a > 0,\ 0 < b < 1, 0 < g < 1)$$

- Endogenous variables: Y, C and G represent national income, consumption and government expenditure, respectively.
- Exogenous variables: I_0 and T_0 represent autonomous investment & autonomous tax respectively.
- 2nd and 3rd equations are behavioral equations showing the behavior of consumption and government expenditure w.r.t income.

$$G = gY \Rightarrow g = \frac{G}{Y}$$

- It implies the government expenditure is a ratio of national income.
- Substituting consumption function and government expenditure function in national income equation.

$$Y = C + I_0 + G$$
$$C = a + b(Y - T_0)$$
$$G = gY$$
$$Y = \{a + b(Y - T_0)\} + I_0 + gY$$
$$Y = (a + bY - bT_0) + I_0 + gY$$
$$Y = a + bY - bT_0 + I_0 + gY$$
$$Y - bY - gY = a - bT_0 + I_0$$
$$Y(1 - b - g) = a - bT_0 + I_0$$
$$Y^* = \frac{a - bT_0 + I_0}{(1 - b - g)}$$

- Parametric restriction for avoiding undefined value of national income.

$$Y^* = \frac{a - bT_0 + I_0}{(1 - b - g)}$$

$$1 - b - g \neq 0$$

$$1 \neq b + g$$

$$b + g \neq 1$$

Lesson 12

USE OF MATRICES IN ECONOMICS

TOPIC 058: MATRICES AND VECTORS

Matrix is a rectangular array of numbers considered as one mathematical object.

- Bold capital letters such as **A**, **B**, … etc. are used to represent a matrix.
- When there are $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns in the matrix, it is a $\boldsymbol{m-by-n}$ matrix (written as $\boldsymbol{m \times n}$).
- A $\boldsymbol{m \times n}$ matrix is of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- $\boldsymbol{a_{11}, a_{12}, \ldots, a_{1n}}$, are the elements/entries of the $\boldsymbol{1^{st}}$ row of matrix.
- $\boldsymbol{a_{21}, a_{22}, \ldots, a_{2n}}$, are the elements/entries of the $\boldsymbol{m^{th}}$ row of matrix.
- $\boldsymbol{a_{11}, a_{21}, \ldots, a_{m1}}$, are the elements/entries of the $\boldsymbol{1^{st}}$ column of matrix.
- $\boldsymbol{a_{1n}, a_{2n}, \ldots, a_{mn}}$, are the elements/entries of the $\boldsymbol{n^{th}}$ column of matrix.

- Alternative ways of writing a matrix are:
- $[a_{ij}]_{m\times n}$ or $(a_{ij})_{m\times n}$
- In more simpler notation:
- $[a_{ij}]$ or (a_{ij})

$$\mathbf{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Zero Matrices:** Four examples w.r.t matrix order

$$[0] \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1×1, 2×2, 4×1 and 4×6

- If matrix appears with either a single row or column, it is known as vector.
- If $\boldsymbol{m} = \mathbf{1}$ and $n > 1$ then it is a row vector.
- If $m > 1$ and $\boldsymbol{n} = \mathbf{1}$ then it is a column vector.

Row vector $\quad \mathbf{a}_{row} = [a_1 \quad a_2 \quad \ldots \quad a_n]$

Column vector $\quad \mathbf{a}_{col} = \begin{bmatrix} a_1 \\ a_2 \\ \ldots \\ a_n \end{bmatrix}$

$$\mathbf{c} = [5 \quad 2 \quad 1 \quad -4]$$

is called a **row vector**,

$$\mathbf{d} = \begin{bmatrix} -3 \\ 10 \\ 6 \\ -7 \\ 1 \\ 9 \\ 2 \end{bmatrix}$$

is called a **column vector**.

TOPIC 059: MATRICES OPERATIONS

Like algebraic expressions, matrices can also be operated upon using arithmetic operators (+, −, ×).

Caveat – I: Matrices can't be divided (÷).

Caveat – II: Matrices can only be solved in linear algebra.

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{C} = \begin{bmatrix} 7+6 & 3+2 & 4+1 \\ 1+0 & 5+4 & 6+4 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \\ 1 & 9 & 10 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ -1 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -5 \\ -1 & -4 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{A} = \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 9 & -3 \\ 4 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{A} = \mathbf{0}$$

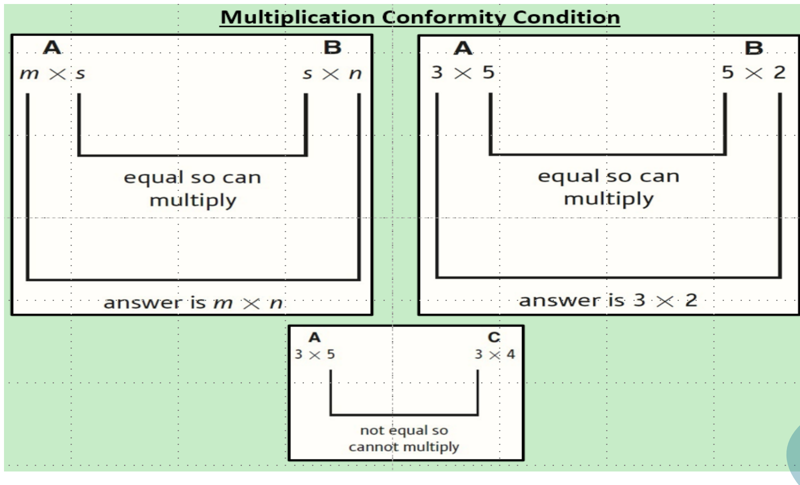
$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

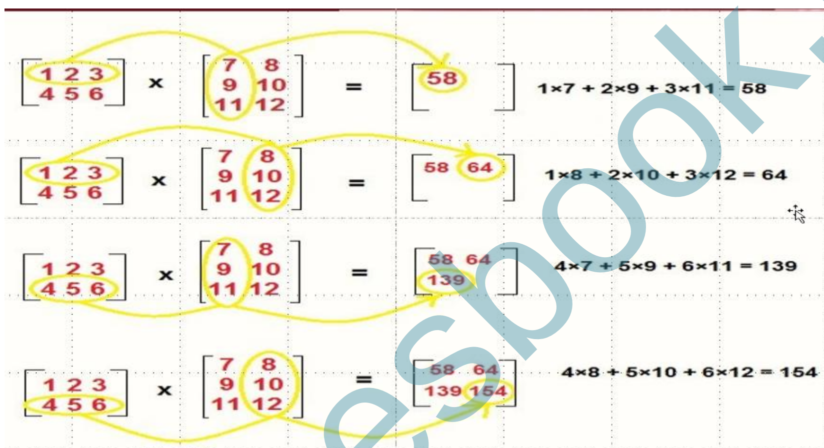
$$\mathbf{B} = \begin{bmatrix} 12 \times 7 & 12 \times 3 & 12 \times 4 \\ 12 \times 1 & 12 \times 5 & 12 \times 6 \end{bmatrix} = \begin{bmatrix} 84 & 36 & 48 \\ 12 & 60 & 72 \end{bmatrix}$$

$$\mathbf{B} = 12\mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad -\mathbf{A} = (-1)\mathbf{A} = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix}$$

$$0\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$





TOPIC 060: USING PRODUCT OF MATRICES TO CALCULATE TOTAL COST

There are three coffee shops in a small college town.
Each coffee shop sells four blends of coffee: 'Espresso', 'Cappuccino', 'Classic' and 'French'.
The cost of coffee for a cup of each blend is:

Cost of Coffee per cup [Cost (C)]	
Expresso	$1.50
Cappuccino	$0.75
Classic	$0.50
French	$1.00

Number of Cups of Coffee Sold Per Week [Quantity Produced (Q)]				
	Expresso	Cappuccino	Classic	French
Coffee Shop 1	112	100	80	35
Coffee Shop 2	182	160	110	58
Coffee Shop 3	206	192	130	76

Total Cost for Each Shop.

Total Cost = Q × C/unit

$$Q = \begin{bmatrix} 112 & 100 & 80 & 35 \\ 182 & 160 & 110 & 58 \\ 206 & 192 & 130 & 76 \end{bmatrix}_{(3\times4)}$$

$$C/unit = C = \begin{bmatrix} 1.5 \\ 0.75 \\ 0.5 \\ 1 \end{bmatrix}_{(4\times1)}$$

since Q [3]×(4), C (4)×[1]: Equal So Can Multiply. Answer is 3×1

$$QC = \begin{bmatrix} 112 & 100 & 80 & 35 \\ 182 & 160 & 110 & 58 \\ 206 & 192 & 130 & 76 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0.75 \\ 0.5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 112\times1.5 + 100\times0.75 + 80\times0.5 + 35\times1 \\ 182\times1.5 + 160\times0.75 + 110\times0.5 + 58\times1 \\ 206\times1.5 + 192\times0.75 + 130\times0.5 + 76\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 318 \\ 506 \\ 594 \end{bmatrix} \Rightarrow \begin{matrix} \text{Shop 1} = \$318 \\ \text{Shop 2} = \$506 \\ \text{Shop 3} = \$594 \end{matrix}$$

Total Costs for Each Shop.

TOPIC 061: USING PRODUCT OF MATRICES TO CALCULATE TOTAL REVENUE AND PROFIT

There are three coffee shops in a small college town.
Each coffee shop sells four blends of coffee: 'Espresso', 'Cappuccino', 'Classic' and 'French'.
The cost of coffee for a cup of each blend is:

Cost of Coffee per cup [Cost (C)]	
Expresso	$1.50
Cappuccino	$0.75
Classic	$0.50
French	$1.00

Number of Cups of Coffee Sold Per Week [Quantity Sold (Q)]				
	Expresso	**Cappuccino**	**Classic**	**French**
Coffee Shop 1	112	100	80	35
Coffee Shop 2	182	160	110	58
Coffee Shop 3	206	192	130	76

Retail Price ($) of Each Blend at Each Shop [Price (P)]				
	Expresso	**Cappuccino**	**Classic**	**French**
Coffee Shop 1	8	4	3	6
Coffee Shop 2	6	2	2	5
Coffee Shop 3	5	3	1	3

Prizes expressed in matrix form:

$$P = \begin{bmatrix} 8 & 4 & 3 & 6 \\ 6 & 2 & 2 & 5 \\ 5 & 3 & 1 & 3 \end{bmatrix}_{(3\times4)} \quad \& \quad Q = \begin{bmatrix} 112 & 100 & 80 & 35 \\ 182 & 160 & 110 & 58 \\ 206 & 192 & 130 & 76 \end{bmatrix}_{(3\times4)}$$

Since $\underset{3\times4}{P}$ $\underset{3\times4}{Q}$

Unequal: Multiplication not possible.

Therefore taking transpose of matrix Q ⇒ Q^t

$$Q^t = \begin{bmatrix} 112 & 182 & 206 \\ 100 & 160 & 192 \\ 80 & 110 & 130 \\ 35 & 58 & 76 \end{bmatrix}$$

Rechecking multiplication pre-requisite:

$\underset{3\times4}{P}$ $\underset{4\times3}{Q^t}$

Equal: Multiplication possible

Answer 3x3

⇒ PQ^t is possible.

$$PQ^t = \begin{bmatrix} 1746 & 2774 & 3262 \\ 1207 & 1922 & 2260 \\ 1045 & 1674 & 1964 \end{bmatrix}_{(3\times3)}$$

Revenue

Shop-1 = 1746

Shop-2 = 1922

Shop-3 = 1964

Total Revenue for Each Shop

$$TR = \begin{bmatrix} 1746 \\ 1922 \\ 1964 \end{bmatrix}$$

$$\pi = TR - TC$$

$$= \begin{bmatrix} 1746 \\ 1922 \\ 1964 \end{bmatrix} - \begin{bmatrix} 318 \\ 506 \\ 594 \end{bmatrix}$$

$$\pi = \begin{bmatrix} 1428 \\ 1416 \\ 1370 \end{bmatrix} \begin{matrix} \text{Shop-1} \\ \text{Shop-2} \\ \text{Shop-3.} \end{matrix}$$

Shop-1 earns the highest profit. (π)

TOPIC 062: QUESTION OF MATRIX DIVISION

Subject to the conformability conditions, matrices, like numbers, can undergo the operations of addition, subtraction, and multiplication.

- However, division of two matrices is not possible i.e. $\boldsymbol{A}/\boldsymbol{B}$ is not possible
- For two numbers $\boldsymbol{a}/\boldsymbol{b}$ is defined when $\boldsymbol{b} \neq \boldsymbol{0}$.
- Alternative representations are $\boldsymbol{a}.\boldsymbol{b}^{-1}$ or $\boldsymbol{b}.\boldsymbol{a}^{-1}$.
- where $\boldsymbol{b}^{-1}$ shows the reciprocal of $\boldsymbol{b}$.
- If $\boldsymbol{AB}^{-1}$ is defined then there is no assurance if $\boldsymbol{B}^{-1}\boldsymbol{A}$ also defined (due to multiplication conformity condition).
- Even if $\boldsymbol{AB}^{-1}$ and $\boldsymbol{B}^{-1}\boldsymbol{A}$ are both defined, still they may not be necessarily equal.
- $\boldsymbol{AB}^{-1}$ and $\boldsymbol{B}^{-1}\boldsymbol{A}$ are two distinct products.
- It is necessary to distinguish between them while specifying.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 5 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \end{bmatrix}$$

$$\frac{A}{B} = ?$$

B → Divisor Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 5 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{1}{30} & \frac{11}{30} & \frac{-1}{3} \\ \frac{-7}{15} & \frac{-2}{15} & \frac{2}{3} \\ \frac{8}{15} & \frac{-2}{15} & \frac{-1}{3} \end{bmatrix}$$

$$A \cdot B^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{30} & \frac{11}{30} & \frac{-1}{3} \\ \frac{-7}{15} & \frac{-2}{15} & \frac{2}{3} \\ \frac{8}{15} & \frac{-2}{15} & \frac{-1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{10} & \frac{-3}{10} & 0 \\ \frac{-3}{10} & \frac{7}{10} & 0 \\ \frac{6}{5} & \frac{1}{5} & -1 \end{bmatrix}$$

Lesson 13

LAWS OF OPERATIONS OF MATRICES

TOPIC 063: COMMUTATIVE, ASSOCIATIVE, AND DISTRIBUTIVE LAWS

Commutative Law in case of addition holds but not in case of multiplication of matrices:

- $A + B = B + A$
- $A.B \neq B.A$

$$AB \neq BA$$

$$Let \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix} \quad then$$

$$AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}, \quad BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

Associative Law holds both in case of addition and multiplication of matrices:

$$A + (B + C) = (A + B) + C$$
$$A(BC) = (AB)C$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}, \qquad (\mathbf{AB})\mathbf{C} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 7 & 5 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} -2 & -1 \\ 7 & 5 \end{pmatrix}, \qquad \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 7 & 5 \end{pmatrix}$$

Thus, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ in this case.

Distributive Law holds in case of Matrices operation:

$$A(B + C) = AB + AC$$

$$\mathbf{B} + \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}, \qquad \mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 6 \\ 5 & 3 \end{pmatrix}$$

and

$$\mathbf{AC} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \mathbf{AB} + \mathbf{AC} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 6 \\ 5 & 3 \end{pmatrix}$$

So $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

TOPIC 064: VECTOR OPERATIONS

Multiplication of Vectors.

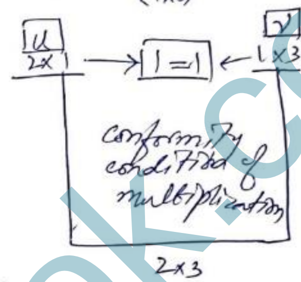
$m \times 1$ a column vector 'u'

$1 \times n$ a row vector 'v'

e.g. $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{(2\times1)}$ and $v = \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}_{(1\times3)}$

$$uv = \begin{bmatrix} 3(1) & 3(4) & 3(5) \\ 2(1) & 2(4) & 2(5) \end{bmatrix}$$

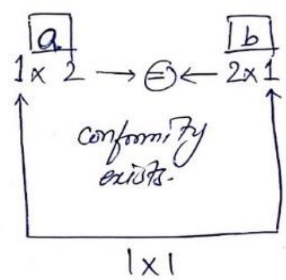
$$= \begin{bmatrix} 3 & 12 & 15 \\ 2 & 8 & 10 \end{bmatrix}$$



e.g. $a = \begin{bmatrix} 3 & 4 \end{bmatrix}_{(1\times2)}$, $b = \begin{bmatrix} 9 \\ 7 \end{bmatrix}_{(2\times1)}$

$$ab = \begin{bmatrix} 3(9) + 4(7) \end{bmatrix}$$

$$ab = \begin{bmatrix} 55 \end{bmatrix}$$



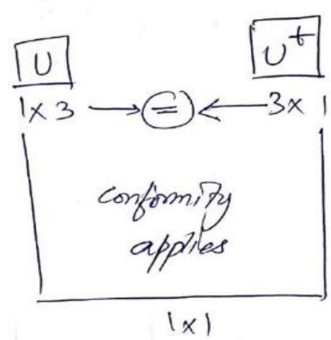
e.g. If $u = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix}_{(1\times3)}$ then find uu^t

$$u^t = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}_{(3\times1)}$$

$$uu^t = \begin{bmatrix} 3\times3 + 6\times6 + 9\times9 \end{bmatrix}$$

$$= \begin{bmatrix} 3^2 + 6^2 + 9^2 \end{bmatrix}$$

uu^t = Sum of squares of elements



Null Vector.

$$0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or}$$

$$0' = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

are referred to as null vectors

e.g. $v_1 = \begin{bmatrix} 6 \\ 21 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 16 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$v_1 - v_2 - v_3 = ?$$

$$= \begin{bmatrix} 6 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ 16 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6-2-4 \\ 21-16-5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \quad \text{(Null vector)}$$

TOPIC 065: TRANSPOSE OF A MATRIX

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$. Find $\mathbf{A}'$ and $\mathbf{B}'$.

$$\mathbf{A}' = \begin{pmatrix} -1 & 2 & 5 \\ 0 & 3 & -1 \end{pmatrix}, \qquad \mathbf{B}' = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}.$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \Longrightarrow \mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

$\mathbf{A}' = (a'_{ij})$, where $a'_{ij} = a_{ji}$

jth row of $\mathbf{A}$ becomes the jth column of $\mathbf{A}'$

ith column of $\mathbf{A}$ becomes the ith row of $\mathbf{A}'$.

RULES FOR TRANSPOSITION

(a) $(\mathbf{A}')' = \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$

(c) $(\alpha\mathbf{A})' = \alpha\mathbf{A}'$

(d) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

TOPIC 066: COFACTORS OF A MATRIX

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \qquad \text{cofactor matrix} = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \quad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \quad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{cases} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ \quad - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{cases}$$

$$|\mathbf{A}| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|\mathbf{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

TOPIC 067: ADJOINT OF A MATRIX

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We first stack the cofactors in their natural positions

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

called the adjugate matrix

Secondly, we take the transpose to get

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

called the adjoint matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

The cofactors of this particular matrix have already been calculated as

$$\begin{array}{lll} A_{11} = 2, & A_{12} = 2, & A_{13} = -2 \\ A_{21} = -11, & A_{22} = 4, & A_{23} = 6 \\ A_{31} = 25, & A_{32} = -10, & A_{33} = -10 \end{array}$$

Stacking these numbers in their natural positions gives the adjugate matrix

$$\begin{bmatrix} 2 & 2 & -2 \\ -11 & 4 & 6 \\ 25 & -10 & -10 \end{bmatrix}$$

The adjoint matrix is found by transposing this to get

$$\begin{bmatrix} 2 & -11 & -25 \\ 2 & 4 & -10 \\ -2 & 6 & -10 \end{bmatrix}$$

Lesson 14

DETERMINANT AND INVERSE OF MATRICES

TOPIC 068: DETERMINANT OF A MATRIX

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

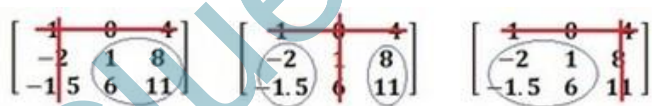
$$\det\begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} = -6 - 1 = -7$$

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of a Matrix

Written in a modulus-sort sign

$$|A| = \begin{vmatrix} 1 & 0 & -4 \\ -2 & 1 & 8 \\ -1.5 & 6 & 11 \end{vmatrix}$$



$$+(1)\begin{vmatrix} 1 & 8 \\ 6 & 11 \end{vmatrix} - (0)\begin{vmatrix} -2 & 8 \\ -1.5 & 11 \end{vmatrix} + (-4)\begin{vmatrix} -2 & 1 \\ -1.5 & 6 \end{vmatrix}$$

$$= 1(11 - 48) - 4(-12 + 1.5) = -37 + 42 = \boxed{5}$$

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{cases} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ \quad - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{cases}$$

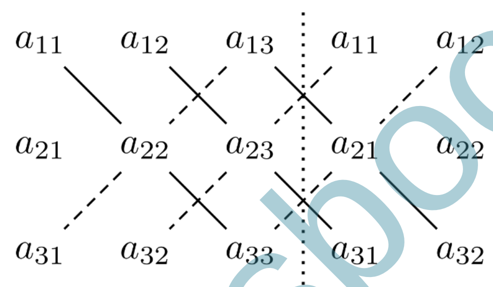
TOPIC 069: SARRUS'S RULE FOR 3X3 ORDER DETERMINANT OF A MATRIX

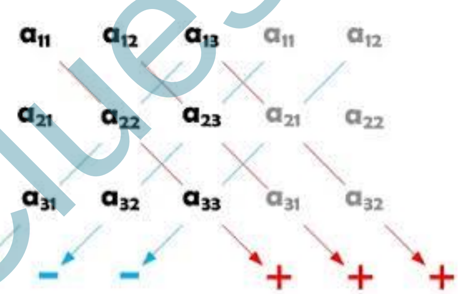
Determinant for 3x3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

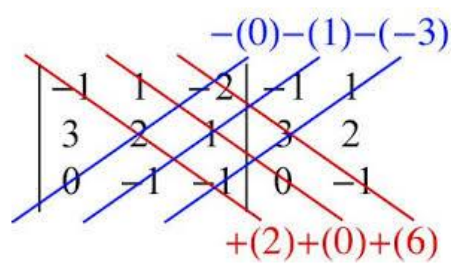
$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}\, a_{22}a_{31}$$





$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}\, a_{22}a_{31}$$



TOPIC 070: INVERSE OF A MATRIX

Example: Find the inverse of A.

$$A = \begin{bmatrix} 2 & 4 \\ -4 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(2)(-10)-(-4)(4)}\begin{bmatrix} -10 & -4 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-4}\begin{bmatrix} -10 & -4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4x6-7x2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \frac{1}{3x8-4x6}\begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{24-24}\begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We first stack the cofactors in their natural positions

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

called the adjugate matrix

Secondly, we take the transpose to get

$$\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

called the adjoint matrix

Finally, we multiply by the scalar

$$\frac{1}{|A|}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

divide each element by the determinant

The last step is impossible if

$$|A| = 0$$

$$\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Suppose that det **A** is nonzero.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Usually called the adjoint of **A**

Three steps in computing above formula
1. for i,j = 1,2,3, replace each a_{ij} by cofactor C_{ij}
2. Take the transpose of the resulting matrix.
3. divide by the determinant of **A**.

Inverse of a Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A' = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\quad AA' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

inverse of A — determinant — Identity matrix

The identity matrix I_n is a n x n square matrix with the main diagonal of 1's and all other elements are 0's.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\text{x}6-7\text{x}2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6 + 7\times-0.2 & 4\times-0.7 + 7\times0.4 \\ 2\times0.6 + 6\times-0.2 & 2\times-0.7 + 6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

TOPIC 071: CONDITION(S) FOR NON-SINGULARITY

Assuming a system of equation.

$$3x + y = 17$$
$$4x - y = 18$$

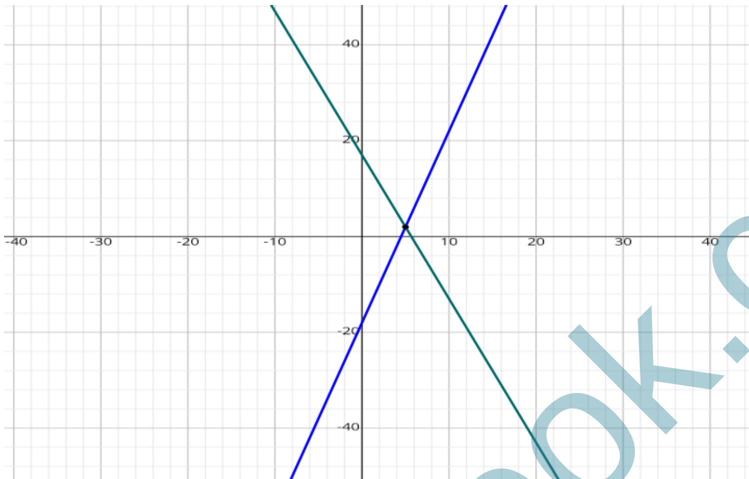
Writing in matrix form

$$\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 18 \end{bmatrix}$$
$$AX = B$$

$A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 17 \\ 18 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = -7 \neq 0$ (A is non-singular matrix & $\exists A^{-1}$)

Solving equations simultaneously
$x = 5,\ y = 2$ [Consistent, Independent]



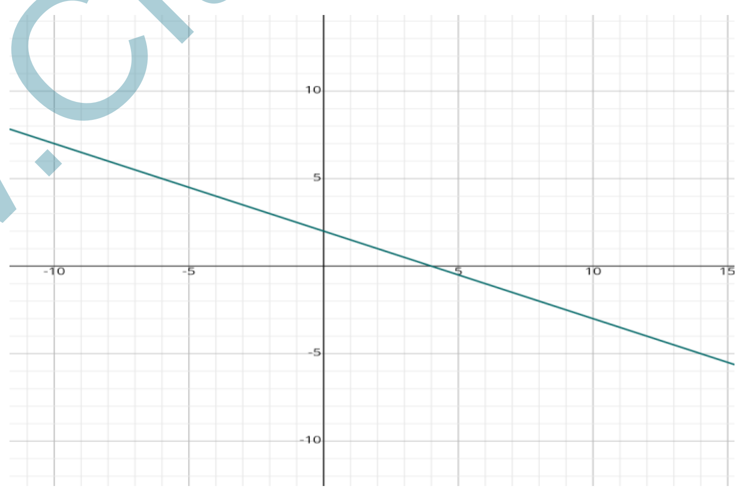
Assume

$2x + 4y = 8$
$x + 2y = 4$

$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0$ (A is a singular matrix & $\nexists A^{-1}$)

Solving equations simultaneously
$x = 2(-y + 2),\ \ y = {}^{1}/_{2}\,(-x + 4)$ **(Undefined: Consistent & Dependent)**
No solution for x and y.



Assume
$3x + 2y = 5$
$6x + 4y = 8$
Matrix form

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$
$$AX = B$$

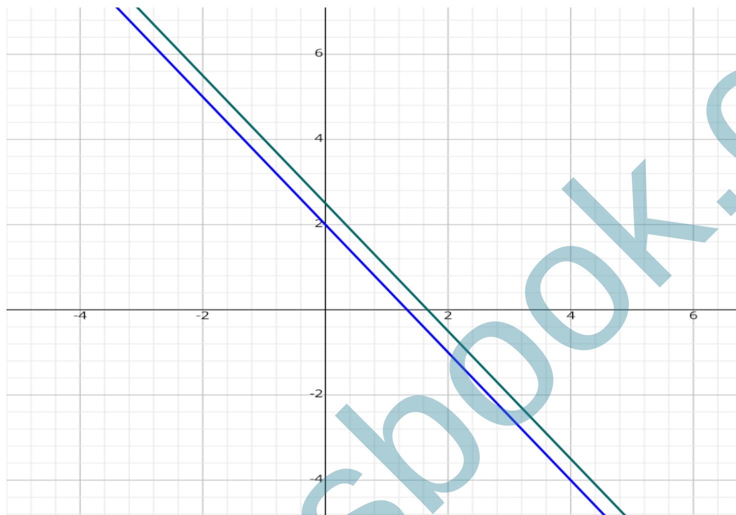
$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ B = $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$|\boldsymbol{A}| = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0$ (A is a Singular matrix & $\nexists A^{-1}$)

Solving equations simultaneously

$2 = 0$ **[Inconsistent, Independent]**

No solution for x and y.



Summarizing conditions:

- **Necessary condition:** Coefficients matrix should be square [Rows= Columns].
- **Sufficient condition:** Determinant of coefficients matrix ≠ 0. Equations should be intersecting. Neither coincident nor parallel.

TOPIC 072: EXPRESSION OF NATIONAL INCOME USING MATRIX FORM

Simple national income model in two endogenous variables Y and C is:

$$\boldsymbol{Y = C + I_o + G_o}$$
$$\boldsymbol{C = a + bY}$$

Re-arranging:

$$\boldsymbol{Y - C = I_o + G_o}$$
$$\boldsymbol{-bY + C = a}$$

Writing in the equation in matrix form.

$\boldsymbol{A}_{(2\times2)} = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}$,

$\boldsymbol{X}_{(2\times1)} = \begin{bmatrix} Y \\ C \end{bmatrix}$

$\boldsymbol{B}_{(2\times1)} = \begin{bmatrix} \boldsymbol{I_o + G_o} \\ a \end{bmatrix}$

Following that:

$$\boldsymbol{AX = B}$$
$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} \boldsymbol{I_o + G_o} \\ a \end{bmatrix}$$
$$\begin{bmatrix} 1(Y) + (-1)(C) \\ (-b)(Y) + 1(C) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I_o + G_o} \\ a \end{bmatrix}$$

$$\begin{bmatrix} Y - C \\ -bY + C \end{bmatrix} = \begin{bmatrix} \boldsymbol{I_o} + \boldsymbol{G_o} \\ a \end{bmatrix}$$

Corresponding elements are equal which is equivalent to original equations.

TOPIC 073: MINORS AND COFACTORS

Determinant, Cofactor, and Minor

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad |M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$|M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$|M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|C_{ij}| \equiv (-1)^{i+j} |M_{ij}|$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$\Rightarrow C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

cofactor of the element (2).

$$\begin{bmatrix} 6 & 2 & 4 \\ 8 & 9 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} 6 & 4 \\ 8 & 3 \end{vmatrix} = -14$$

The cofactor is $(-1)^{2+3}(-14) = (-1)(-14) = 14.$

Lesson 15

MATRIX INVERSION METHOD

TOPIC 074: MARKET MODEL ANALYSIS USING MATRIX INVERSION METHOD

$$Q_d = Q_s$$

$$Q_d = a - bP$$

$$Q_s = -c + dP$$

$$Ax = b \Leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & d \end{bmatrix} \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

$$det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & d \end{bmatrix} = -(d+b) \neq 0$$

$$x = A^{-1}b$$

$$x = A^{-1}b \Leftrightarrow \frac{1}{-(d+b)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

$$\frac{1}{-(d+b)} \begin{bmatrix} -(ad-bc) \\ -(ad-bc) \\ -(a+c) \end{bmatrix} = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}$$

TOPIC 075: NATIONAL INCOME ANALYSIS USING MATRIX INVERSION METHOD

Consider National income model.

$$Y = C + I_0 + G_0 \quad , \quad C = a + bY.$$

Rearranging for matrix form

$$Y - C = I_0 + G_0 \quad , \quad -bY + C = a$$

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}, \quad B = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix} \; \& \; X = \begin{bmatrix} Y \\ C \end{bmatrix}$$

$$\Rightarrow \quad AX = B.$$

$$\Rightarrow \quad X = A^{-1}B$$

$$A^{-1} = \frac{\text{adj}.A}{|A|}$$

$$|A| = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = 1 - b \neq 0 \text{ if } b \neq 1$$

$$\text{adj}.A = \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

$$\begin{bmatrix} Y \\ C \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} 1 \times (I_0 + G_0) + 1(a) \\ b \times (I_0 + G_0) + 1(a) \end{bmatrix} = \left(\frac{1}{1-b}\right)\begin{bmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} Y^* &= \frac{1}{1-b}(I_0 + G_0 + a) \quad \& \\ C^* &= \frac{1}{1-b}\left(b(I_0 + G_0) + a\right) \end{aligned}}$$

Equilibrium Y & C using matrix inversion method

TOPIC 076: EQUILIBRIUM PRICES USING MATRIX INVERSION METHOD

equilibrium condition for three related markets is given by

$$11P_1 - P_2 - P_3 = 31$$
$$-P_1 + 6P_2 - 2P_3 = 26$$
$$-P_1 - 2P_2 + 7P_3 = 24$$

equilibrium price for each market.

$$\begin{bmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 26 \\ 24 \end{bmatrix}$$

$$|A| = 11(38) + 1(-9) - 1(8) = 401.$$

$$C = \begin{bmatrix} \begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ -1 & 7 \end{vmatrix} & \begin{vmatrix} -1 & 6 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & -1 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 11 & -1 \\ -1 & 7 \end{vmatrix} & -\begin{vmatrix} 11 & -1 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} & -\begin{vmatrix} 11 & -1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 11 & -1 \\ -1 & 6 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix}$$

$$A^{-1} = \frac{1}{401}\begin{bmatrix} 38 & 9 & 8 \\ 9 & 76 & 23 \\ 8 & 23 & 65 \end{bmatrix} = \begin{bmatrix} \frac{38}{401} & \frac{9}{401} & \frac{8}{401} \\ \frac{9}{401} & \frac{76}{401} & \frac{23}{401} \\ \frac{8}{401} & \frac{23}{401} & \frac{65}{401} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{38}{401} & \frac{9}{401} & \frac{8}{401} \\ \frac{9}{401} & \frac{76}{401} & \frac{23}{401} \\ \frac{8}{401} & \frac{23}{401} & \frac{65}{401} \end{bmatrix}\begin{bmatrix} 31 \\ 26 \\ 24 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1178 + 234 + 192}{401} \\ \frac{279 + 1976 + 552}{401} \\ \frac{248 + 598 + 1560}{401} \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{bmatrix}$$

Lesson 16

CRAMER'S RULE IN MATRICES

TOPIC 077: SOLVING MARKET MODEL USING CRAMER'S RULE

Consider a linear partial market model:

$$Q_d = a + bP; Q_s = c + dP; Q_d = Q_s.$$

$$\bar{Q} = a + b\bar{P} \rightarrow \bar{Q} - b\bar{P} = a$$
$$\bar{Q} = c + d\bar{P} \rightarrow \bar{Q} - d\bar{P} = c$$

$$\begin{pmatrix} 1 & -b \\ 1 & -d \end{pmatrix}\begin{pmatrix} \bar{Q} \\ \bar{P} \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\bar{Q} = \frac{\begin{vmatrix} a & -b \\ c & -d \end{vmatrix}}{\begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}}$$

$$= \frac{-ad + bc}{-d + b} = \frac{bc - ad}{b - d}$$

$$\bar{P} = \frac{\begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix}}{\begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}}$$

$$= \frac{c - a}{-d + b} = \frac{a - c}{b - d}$$

TOPIC 078: EQUILIBRIUM PRICES USING CRAMER'S RULE

$$11p_1 - p_2 - p_3 = 31$$
$$-p_1 + 6p_2 - 2p_3 = 26$$
$$-p_1 - 2p_2 + 7p_3 = 24$$

$$|A| = \begin{vmatrix} 11 & -1 & -1 \\ -1 & 6 & -2 \\ -1 & -2 & 7 \end{vmatrix} = 11(38) + 1(-9) - 1(8) = 401$$

$$|A_1| = \begin{vmatrix} 31 & -1 & -1 \\ 26 & 6 & -2 \\ 24 & -2 & 7 \end{vmatrix} = 31(38) + 1(230) - 1(-196) = 1604$$

$$|A_2| = \begin{vmatrix} 11 & 31 & -1 \\ -1 & 26 & -2 \\ -1 & 24 & 7 \end{vmatrix} = 11(230) - 31(-9) - 1(2) = 2807$$

$$|A_3| = \begin{vmatrix} 11 & -1 & 31 \\ -1 & 6 & 26 \\ -1 & -2 & 24 \end{vmatrix} = 11(196) + 1(2) + 31(8) = 2406$$

$$\bar{p}_1 = \frac{|A_1|}{|A|} = \frac{1604}{401} = 4 \qquad \bar{p}_2 = \frac{|A_2|}{|A|} = \frac{2807}{401} = 7 \qquad \bar{p}_3 = \frac{|A_3|}{|A|} = \frac{2406}{401} = 6$$

TOPIC 79: NATIONAL INCOME DETERMINATION USING CRAMER'S RULE

2-sector national income model:

$$Y - C = I_0 + G_0$$
$$-bY + C = a$$

Then Cramer's rule yields

$$Y = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}}$$

$$= \frac{a + I_0 + G_0}{1 - b},$$

$$C = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}}$$

$$= \frac{a + b(I_0 + G_0)}{1 - b}$$

Lesson 17

INPUT OUTPUT ANALYSIS USING MATRICES

TOPIC 080: INPUT COEFFICIENT MATRIX

Attributed to Wassely Leontief (1951)

- Static version: "What level of output should each of the n industries in an economy produce, in order that it will just be sufficient to satisfy the total demand for that product?"
- Input-Output: output of one industry is needed as input in another industry & vice versa – Inter-industry dependence.
- Input-output analysis can be of great use economic planning.
- Need for 'correct' level of output based on technical input-output, rather than market equilibrium conditions.
- Mathematically speaking, it's the solution of simultaneous equations.
- Normally has large number of industries.

Assumptions

- Each industry has homogeneous output or jointly product in fixed proportions.
- Each industry has fixed input ratio for output.
- CRS in each industry: **k-fold ↑ in inputs ⇒ k-fold ↑ in output.**
- **Input coefficient matrix** contains **input coefficient.**

$$Input\ Coefficient = \frac{Input\ in\ PKR}{Output\ in\ PKR}$$

$$\boldsymbol{a_{ij}} = \frac{\boldsymbol{Input\ i\ in\ PKR}}{\boldsymbol{Output\ j\ in\ PKR}}$$

$$a_{32} = 0.35 = \frac{0.35}{1}$$

0.35 PKR input **'3' (2nd input)** is needed in producing 1 PKR of **'2' (2nd output).**

Input	Output I	II	III	.	.	.	N
I	a_{11}	a_{12}	a_{13}	.	.	.	a_{1n}
II	a_{21}	a_{22}	a_{23}	.	.	.	a_{2n}
III	a_{31}	a_{32}	a_{33}	.	.	.	a_{3n}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
N	a_{n1}	a_{n2}	a_{n3}	.	.	.	a_{nn}

- Number of input coefficients in input coefficient matrix are:
- $= Number\ of\ Inputs \times Number\ of\ Outputs$
- Zero elements in principal diagonal:

Input	Output: I	II	III	.	.	.	N
I	a_{11}	a_{12}	a_{13}	.	.	.	a_{1n}
II	a_{21}	a_{22}	a_{23}	.	.	.	a_{2n}
III	a_{31}	a_{32}	a_{33}	.	.	.	a_{3n}
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
N	a_{n1}	a_{n2}	a_{n3}	.	.	.	a_{nn}

input output coefficient matrix for a five sector economy.

	Food	Housing	Basic materials	Energy	Manufactured	Services
Food	0.10	0.03	0.04	0.01	0.02	0.01
Housing	0.05	0.20	0.06	0.05	0.03	0.20
Basic Materials	0.02	0.07	0.30	0.07	0.04	0.01
Energy	0.01	0.06	0.09	0.22	0.02	0.07
Manufactured	0.02	0.03	0.03	0.13	0.17	0.05
Services	0.04	0.02	0.02	0.05	0.06	0.25

TOPIC 081: ECONOMIC MEANING OF HAWKINS-SIMON CONDITION

Attributed to David Hawkins and Herbert A. Simon,

- Guarantees the existence of a non-negative output vector.
- Assume a 2-sector economy:

  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

  $(I - A) = B = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}$

- **Two conditions**

$$|B_1| > 0 \;\&\; |B_2| > 0$$

- **First condition**

$$|B_1| = |1 - a_{11}| > 0$$
$$a_{11} < 1$$

$a_{11} < 1$ implies that amount of first commodity used in production of 1st commodity is less than PKR 1.

- **Second condition**

$$|B_2| = \begin{vmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{vmatrix} > 0$$

$$= (1 - a_{11})(1 - a_{22}) - a_{12}a_{21}$$
$$= 1 + a_{11}a_{22} - a_{11} - a_{22} + a_{12}a_{21} > 0$$

$$= 1 > a_{11} + a_{22} - a_{11}a_{22} + a_{12}a_{21}$$
$$= 1 > a_{11} + a_{12}a_{21} + a_{22} - a_{11}a_{22}$$
$$= 1 > a_{11} + a_{12}a_{21} + a_{22}(1 - a_{11})$$
$$= a_{11} + a_{12}a_{21} + a_{22}(1 - a_{11}) < 1$$

$a_{22}(1 - a_{11})$ can be omitted without affecting the inequality.

$$= a_{11} + a_{12}a_{21} < 1$$
$$= \underbrace{a_{11}}_{\substack{Direct \\ Use\ of\ 1}} + \underbrace{a_{12}a_{21}}_{\substack{Indirect \\ Use\ of\ 1}} < 1$$

- Practicability and viability in production via Hawkins-Simon condition.

TOPIC 082: INPUT-OUTPUT ANALYSIS IN CASE OF OPEN ECONOMY

Assume a 3-sector economy (Agriculture, Industry & Services).
Its input coefficient matrix is:

	Agri	Indu	Serv
Agri	0.2	0.3	0.2
Indu	0.4	0.1	0.2
Serv	0.1	0.3	0.2

- Agriculture uses PKR 0.2 of agricultural output in producing PKR 1.
- Agriculture uses PKR 0.4 & PKR 0.1 of industrial & services outputs, respectively, in producing PKR 1.
- In addition to mutual demand, consumer demand also exists.
- Demands (in billion PKR) are in demand vector.

$$D = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$$

- Forming a system of equations:

- Total demand = Mutual demand + Consumer demand

$$x = \underbrace{0.2x + 0.3y + 0.2z}_{Mutual\ demand} + \underbrace{10}_{\substack{Consumer \\ Demand}}$$

$$y = \underbrace{0.4x + 0.1y + 0.2z}_{Mutual\ demand} + \underbrace{5}_{\substack{Consumer \\ Demand}}$$

$$z = \underbrace{0.1x + 0.3y + 0.2z}_{Mutual\ demand} + \underbrace{6}_{\substack{Consumer \\ Demand}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$$

$$X = AX + D$$
$$X - AX = D$$
$$X(I - A) = D$$

$$X = (I - A)^{-1}D$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

$$(I - A) = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \text{(Leontief Matrix)}$$

$$|I - A| = 0.384$$

$$adj.\ (I - A) = \begin{vmatrix} 0.66 & 0.30 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.60 \end{vmatrix}$$

$$(I - A)^{-1} = \frac{adj.(I - A)}{|I - A|}$$

$$(I - A)^{-1} = \frac{1}{0.384} \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix}$$

$$X = (I - A)^{-1}D$$

$$X = \frac{1}{0.384} \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}$$

Interpretation: To suffice for mutual and consumer demand.

- Agricultural output = 24.84 billion PKR.
- Industrial output = 20.68 billion PKR.
- Services output = 18.36 billion PKR.

TOPIC 083: INPUT-OUTPUT ANALYSIS IN CASE OF CLOSED ECONOMY

Assume a 3-sector economy (Agriculture, Industry & Services).
Its input coefficient matrix is:

	Agri	Indu	Serv
Agri	0.2	0.3	0.2
Indu	0.2	0.6	0.4
Serv	0.6	0.1	0.4

- No consumer demand:

$$\text{D} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- All outputs are consumed in process of production.
- All columns sum to 1.

Total demand = Inter-industry demand

$$x = \underbrace{0.2x + 0.3y + 0.2z}_{inter-indutry\ demand}$$

$$y = \underbrace{0.2x + 0.6y + 0.4z}_{inter-indutry\ demand}$$

$$z = \underbrace{0.6x + 0.1y + 0.4z}_{inter-indutry\ demand}$$

Writing in matrix form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.2 & 0.6 & 0.4 \\ 0.6 & 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

However, the homogeneous system of equations can be solved simultaneously:

$$x = \frac{10}{13}z,\ y = \frac{18}{13}z, z$$

$$x = \frac{10}{13}z,\ y = \frac{18}{13}z, z$$

In ratio form (in terms of z):

$$\frac{10}{13} : \frac{18}{13} : 1$$

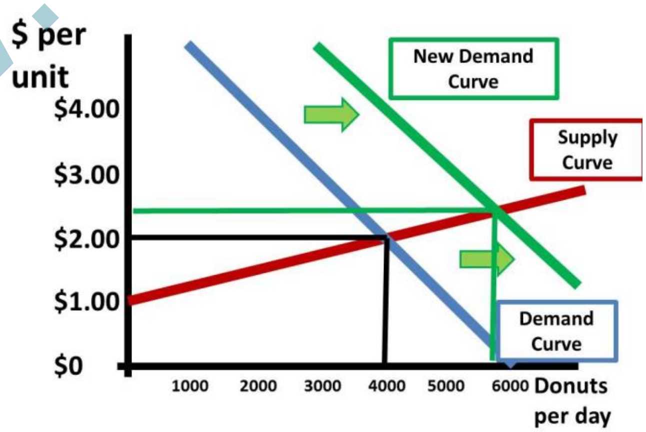
Converting to whole numbers:

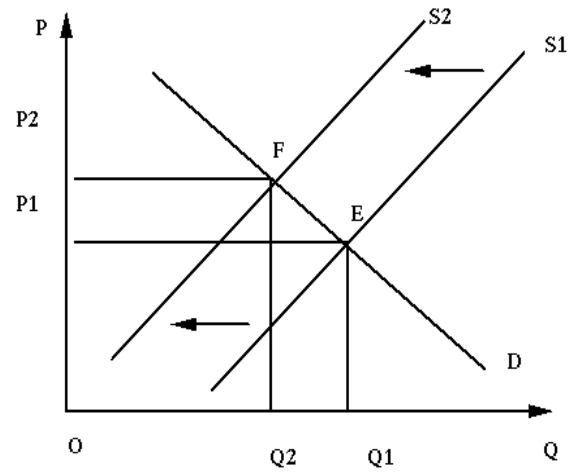
$$\mathbf{10 : 18 : 13}$$

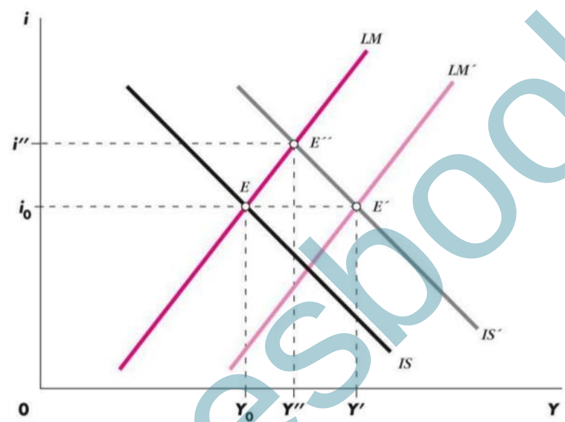
For the closed model of 3 industries, we get a guiding ratio.

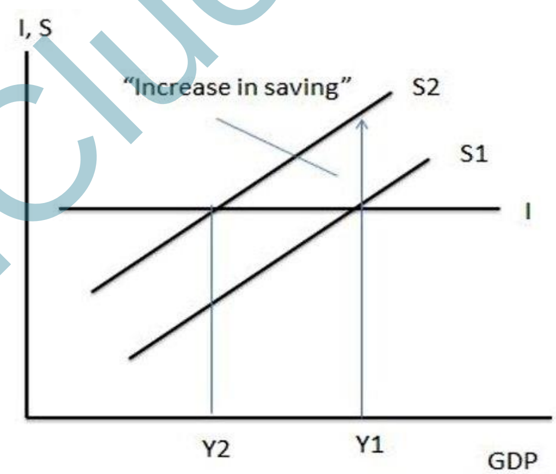
TOPIC 084: THE NEED AND NATURE OF COMPARATIVE STATICS

If all variables are at rest, an equilibrium is often called a static.
Comparing equilibria is, therefore, called comparative statics.
How would be the new equilibrium compared with the old?
Qualitative (in which direction) and quantitative (by how much) aspects.









Two Assumptions

- System returns to equilibrium instantaneously – merely compare the initial (pre-change) equilibrium (E_1) with the final (post-change) equilibrium (E_2).
- Stable equilibrium.

- "Rate of change of the equilibrium value of an endogenous variable w.r.t the change in a particular parameter or exogenous variable – Need for derivatives."
- Functionally speaking:

$$\boldsymbol{y = f(x)}$$

y = Equilibrium value of endogenous variable
x = parameter/exogenous variable.

Difference Quotient

- Let, Δ = change in x from x_0 to x_1.
- Then, $\Delta x = x_1 - x_0$.
  x_0 = old value of x
  $x_0 + \Delta x$ = new value of x

- Then function $y = f(x)$ changes from $y = f(x_0)$ to $f(x_0 + \Delta x)$.
- Change in y per unit of change in x: $\frac{\Delta y}{\Delta x}$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Numerical

$y = f(x) = 3x^2 - 4$

$$f(x_0) = 3x_0^2 - 4, f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4 \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \frac{3(x_0 + \Delta x)^2 - 4 - 3x_0^2 + 4}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = 6x_0 + 3\Delta x$$

If $\Delta x \to 0$, and $\frac{\Delta y}{\Delta x}$ exists. Expression $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right)$ is derivative of the function.

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = f'(x)$$
$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}(6x + 3\Delta x) \cong 6x = \frac{dy}{dx}$$

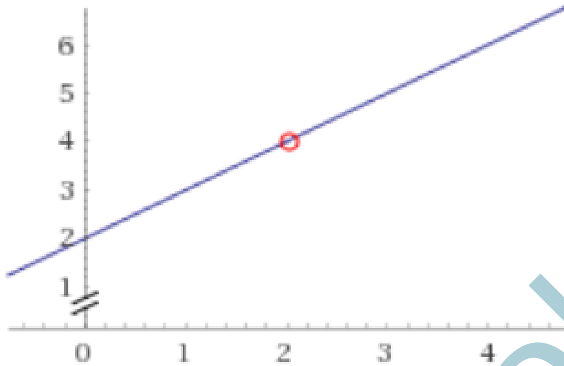
Lesson 18

CONCEPT OF DERIVATIVE AND RULES OF DIFFERENTIATION

TOPIC 085: CONCEPT OF LIMIT AND CONTINUITY

Limit: "A point or level beyond which something does not or may not extend or pass."

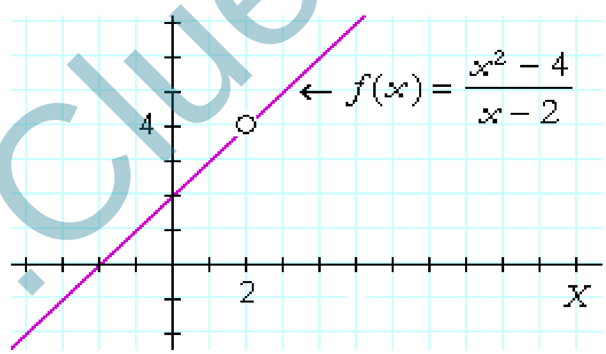
Let $y = \frac{x^2-4}{x-2}$

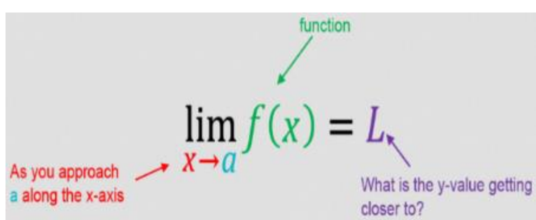


Clearly, $y = \infty$ at $x = 2$. i.e. $y(2) = 0$.
Before and after x =2, y is defined.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	3.9	3.99	3.999	∞	4.001	4.01	4.1

Observe: x approaches to 2, f(x) approaches close to 4. i.e. $x \to 2$, $f(x) \to 4$.





$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$

Factor x^2-4: $(x+2)(x-2)$

$= \frac{(x+2)(x-2)}{x-2}$

Cancel the common factor: $x-2$

$= x+2$

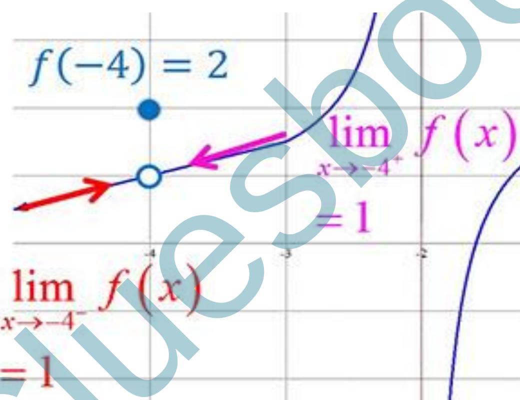
$= \lim_{x\to 2}(x+2)$

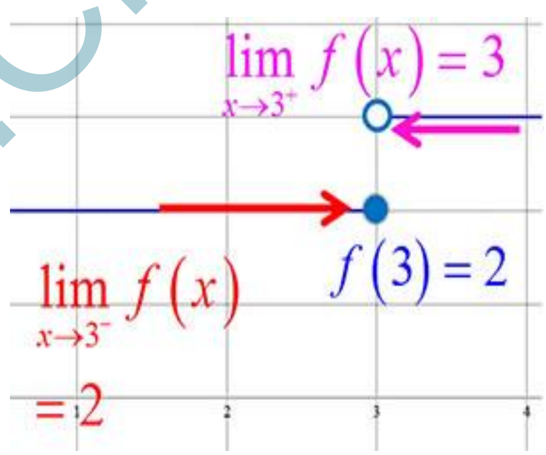
Plug in the value $x=2$

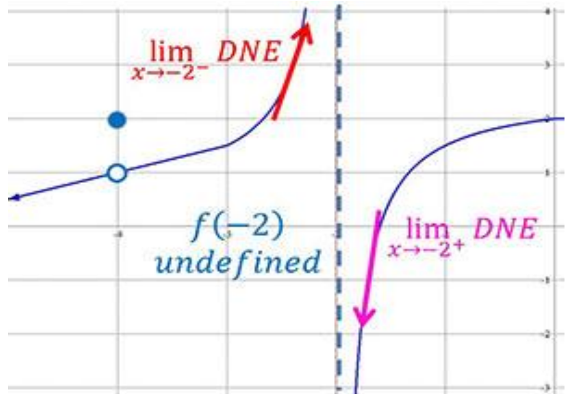
$= 2+2$

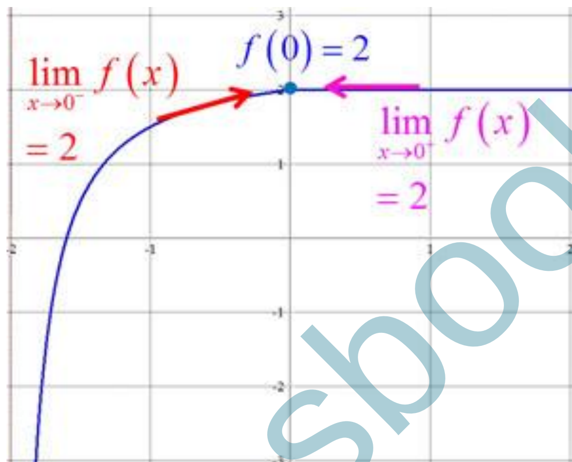
$= 4$

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = 4$$









Let $y(x) = \frac{4x+10}{x^2-2x-15}$

- For x = 3 or x = 5, denominator=0, $y(x) = \infty$
- Function $y(x)$ is discontinuous at x = 3 and x = 5.

$$\lim_{x \to 5} \left(\frac{4x + 10}{x^2 - 2x - 15} \right)$$

Indeterminate

$$\lim_{x \to -3} \left(\frac{4x + 10}{x^2 - 2x - 15} \right)$$

Indeterminate

$$\lim_{x \to 3} \left(\frac{4x + 10}{x^2 - 2x - 15} \right)$$

$$-1.8333333333333$$

$$\lim_{x \to -5} \left(\frac{4x + 10}{x^2 - 2x - 15} \right)$$

$$-0.5$$

TOPIC 086: RATE OF CHANGE, SLOPE & DERIVATIVE

For any continuous function $\{y = f(x)\}$, rate of change can be calculated using differentiation.

- Differentiation is process of calculating derivatives.
- Derivative is also the slope of the function.
- Derivative of function $y = \frac{d}{dx}(y)$ a.k.a Leibniz's notation of derivative
  d represents the change 'Δ'.
  Rate of change is function $y = f(x)$ is $\frac{\Delta y}{\Delta x}$.
  Therefore; $\frac{dy}{dx} \cong \frac{\Delta y}{\Delta x}$
  i.e. Derivative of function represents the rate of change.
- Slope also shows the rate of change.

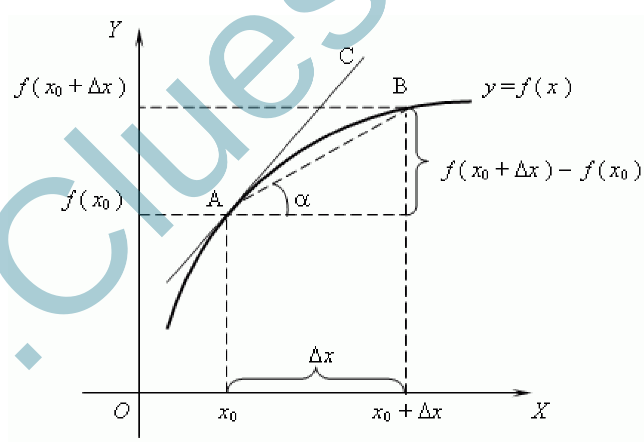
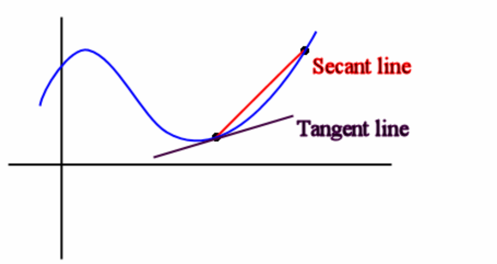
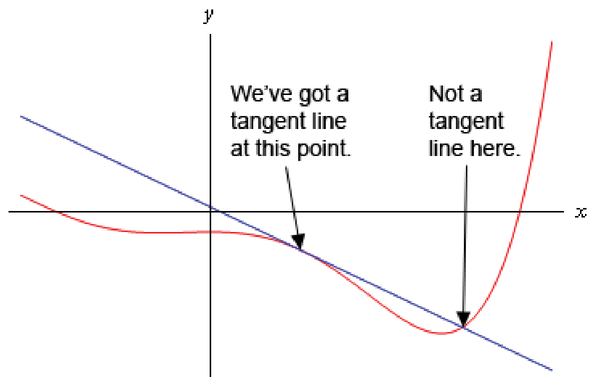
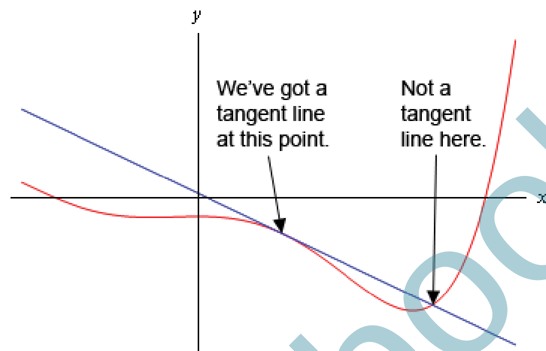


Fig. 1







Therefore, derivative, rate of change and slope can be used synonymously.

$$\frac{dy}{dx} \cong \frac{\Delta y}{\Delta x} \cong slope\ of\ y = f(x)$$

- In economics, taking derivative is equal to calculating the marginal function of the original function.

e.g. Derivative of total cost function gives marginal cost (rate of change of total cost).

Lesson 19

PRODUCT RULE AND QUOTIENT RULE OF DIFFERENTIATION

TOPIC 087: DIFFERENTIATION RULES FOR SINGLE VARIABLE FUNCTIONS: CONSTANT FUNCTION RULE AND POWER FUNCTION RULE

Rules of differentiation are necessary for calculating derivatives.

- Most commonly used are:
  - Constant function rule
  - Power rule
  - Sum-difference rule
  - Product rule
  - Quotient rule

POWER RULE

$$f(x) = x^a \Longrightarrow f'(x) = ax^{a-1}$$

$$y = \frac{x^{100}}{100} = \frac{1}{100}x^{100} \Longrightarrow y' = \frac{1}{100}100x^{100-1} = x^{99}$$

(a) $\frac{d}{dx}\left(x^{-0.33}\right) = -0.33x^{-0.33-1} = -0.33x^{-1.33}$

(b) $\frac{d}{dr}(-5r^{-3}) = (-5)(-3)r^{-3-1} = 15r^{-4}$

(c) $\frac{d}{dp}(Ap^{\alpha} + B) = A\alpha p^{\alpha-1}$

(d) $\frac{d}{dx}\left(\frac{A}{\sqrt{x}}\right) = \frac{d}{dx}(Ax^{-1/2}) = A\left(-\frac{1}{2}\right)x^{-1/2-1} = -\frac{A}{2}x^{-3/2} = \frac{-A}{2x\sqrt{x}}$

$$\frac{d}{dx}A = 0$$

$$\frac{d}{dx}[A + f(x)] = \frac{d}{dx}f(x)$$

$$\frac{d}{dx}[Af(x)] = A\frac{d}{dx}f(x)$$

TOPIC 088: SUM-DIFFERENCE RULE OF DIFFERENTIATION

DIFFERENTIATION OF SUMS AND DIFFERENCES

If both f and g are differentiable at x, then the sum $f + g$ and the difference $f - g$ are both differentiable at x, and

$$F(x) = f(x) \pm g(x) \implies F'(x) = f'(x) \pm g'(x)$$

In Leibniz's notation:

$$\frac{d}{dx}\big(f(x) \pm g(x)\big) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

$$\tfrac{d}{dx}\big(3x^8 + x^{100}/100\big) = \tfrac{d}{dx}(3x^8) + \tfrac{d}{dx}\big(x^{100}/100\big) = 24x^7 + x^{99}$$

$$\frac{d}{dx}(f(x) - g(x) + h(x)) = f'(x) - g'(x) + h'(x)$$

TOPIC 089: SUM-DIFFERENCE RULE: NUMERICAL ANALYSIS OF COST FUNCTION

$$C = Q^3 - 4Q^2 + 10Q + 75$$

- Is a cost function in cubic form.
- To calculate marginal cost, we can take its derivate with respect to Q.

$$\frac{d}{dQ}(C) = \frac{d}{dQ}(Q^3 - 4Q^2 + 10Q + 75)$$

$$\frac{d}{dx}(f(x) - g(x) + h(x)) = f'(x) - g'(x) + h'(x)$$

$$\frac{d}{dQ}(C) = \frac{d}{dQ}(Q^3) - \frac{d}{dQ}(4Q^2) + \frac{d}{dQ}(10Q) + \frac{d}{dQ}(75)$$

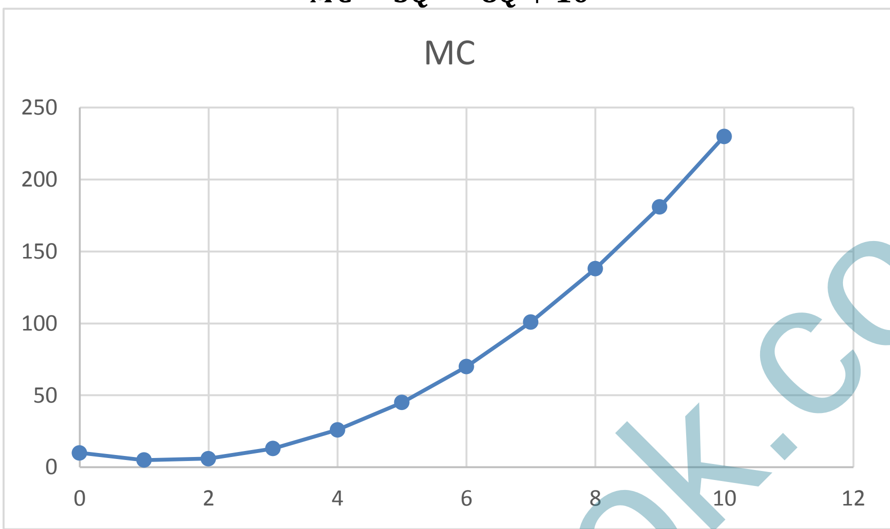
$$\frac{d}{dQ}(C) = MC = C^{(1)} = 3Q^2 - 8Q + 10 + 0$$

$$MC = 3Q^2 - 8Q + 10$$

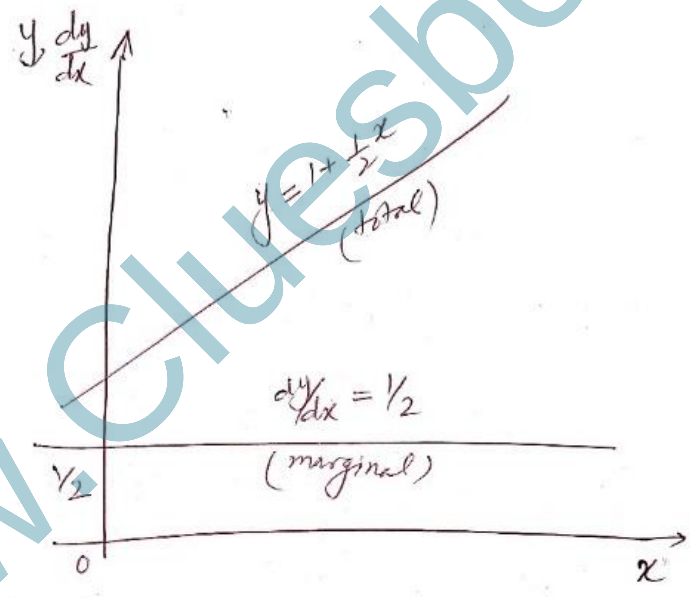
Slope of Cost Curve, Rate of Change in Cost Function, Marginal Cost

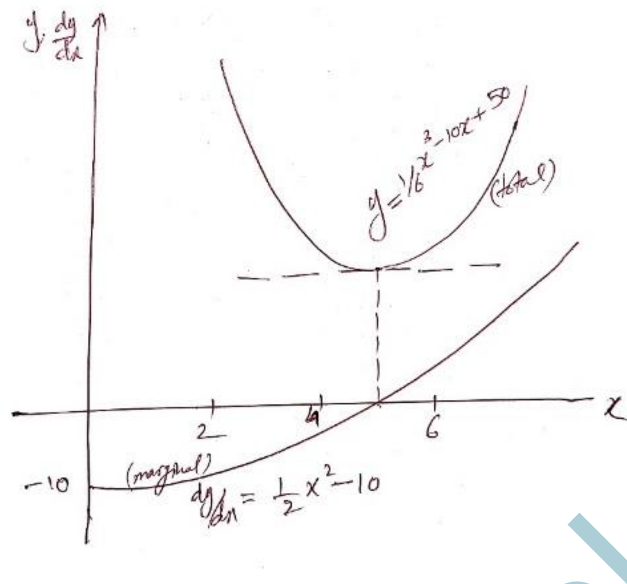
Q	MC
0	10
1	5
2	6
3	13
4	26
5	45
6	70
7	101
8	138
9	181
10	230

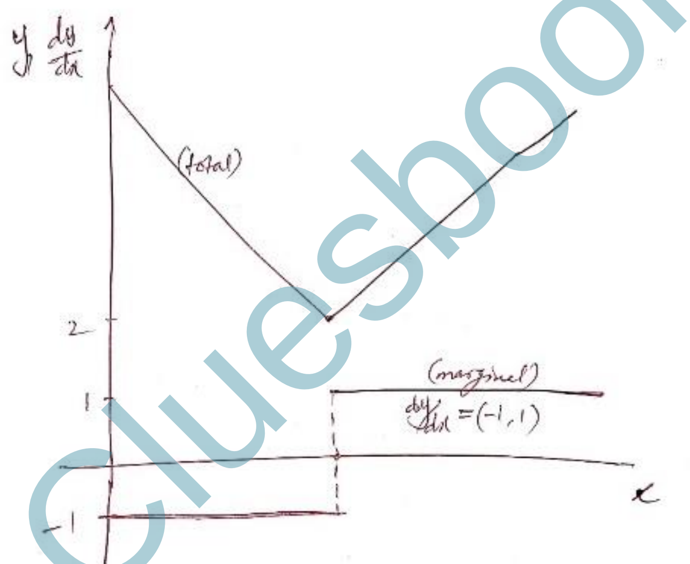
$$MC = 3Q^2 - 8Q + 10$$



TOPIC 090: UNDERSTANDING GRAPHS OF FUNCTION AND THEIR DERIVATIVES




(total)
(marginal)
-10
2
4
6
x


(total)
(marginal)
2
1
-1
x

Lesson 20

COST AND REVENUE ANALYSIS USING DIFFERENTIATION

TOPIC 091: PRODUCT RULE OF DIFFERENTIATION

THE DERIVATIVE OF A PRODUCT

If both f and g are differentiable at the point x, then so is $F = f \cdot g$, and

$$F(x) = f(x) \cdot g(x) \Longrightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

In Leibniz's notation, the product rule is expressed as:

$$\frac{d}{dx}[f(x) \cdot g(x)] = [\frac{d}{dx} f(x)] \cdot g(x) + f(x) \cdot [\frac{d}{dx} g(x)]$$

find $h'(x)$ when $h(x) = (x^3 - x) \cdot (5x^4 + x^2)$.

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Here $f(x) = (x^3 - x)$ & $g(x) = (5x^4 + x^2)$

$\Rightarrow f'(x) = \frac{d}{dx}(x^3 - x)$ & $g'(x) = \frac{d}{dx}(5x^4 + x^2)$

$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x)$

$= 5\frac{d}{dx}(x^4) + \frac{d}{dx}(x^2)$

$f'(x) = (3x^2 - 1)$ $g'(x) = (20x^3 + 2x)$

$$= (3x^2 - 1) \cdot (5x^4 + x^2) + (x^3 - x) \cdot (20x^3 + 2x)$$

$= (15x^6 + 3x^4 - 5x^4 - x^2) + (20x^6 + 2x^4 - 20x^4 - 2x^2)$

$= 15x^6 + 20x^6 + 3x^4 - 5x^4 + 2x^4 - 20x^4 - x^2 - 2x^2$

$$h'(x) = 35x^6 - 20x^4 - 3x^2$$

TOPIC 092: RELATIONSHIP BETWEEN AVERAGE REVENUE AND MARGINAL REVENUE USING PRODUCT RULE

Given an average revenue function:

$$AR = 15 - Q$$

To find marginal revenue function, one needs to find total revenue first.

$$\frac{TR}{Q} = AR \;\Rightarrow\; TR = AR.Q$$
$$TR = (15 - Q).Q$$
$$TR = 15Q - Q^2$$

Differentiating the total revenue function w.r.t Q, we get marginal revenue.

$$MR = \frac{d(TR)}{dQ}$$
$$= \frac{d(15 - Q^2)}{dQ}$$
$$MR = 15 - 2Q$$

Symbolic treatment of revenue function:

$$R = AR.Q$$

$R = f(Q).Q$ *[as AR depends on quantity sold]*

$$MR = \frac{dR}{dQ}$$

Differentiating w.r.t Q using product rule

$h'(x) = f'(x).g(x) + g'(x).f(x)$
Here $f(x) = f(Q)$ and $g(x) = Q$

$$MR = f(Q).\left(\frac{dQ}{dQ}\right) + Q.\left[\frac{d\{f(Q)\}}{dQ}\right]$$
$$MR = f(Q).(1) + Q.\{f'(Q)\}$$
$$MR = f(Q) + Q.f'(Q)$$

TOPIC 093: QUOTIENT RULE OF DIFFERENTIATION

THE DERIVATIVE OF A QUOTIENT

If f and g are differentiable at x and $g(x) \neq 0$, then $F = f/g$ is differentiable at x, and

$$F(x) = \frac{f(x)}{g(x)} \implies F'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Compute $F'(x)$ and $F'(4)$ when $F(x) = \dfrac{3x-5}{x-2}$.

Here $f(x) = (3x-5)$ & $g(x) = (x-2)$

$\Rightarrow f'(x) = \frac{d}{dx}(3x-5)$ & $g'(x) = \frac{d}{dx}(x) - 0$

$= 3\left(\frac{dx}{dx}\right)$ $\qquad \boxed{g'(x) = 1}$

$= 3(1)$

$\boxed{f'(x) = 3}$

$$F'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

$$F'(x) = \frac{3\cdot(x-2) - (3x-5)\cdot 1}{(x-2)^2} = \frac{3x-6-3x+5}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

To find $F'(4)$, we put $x = 4$ in the formula for $F'(x)$ to get $F'(4) = -1/(4-2)^2 = -1/4$.

TOPIC 094: MARGINAL PROPENSITY TO CONSUME VIA DIFFERENTIATION WITH AND WITHOUT TAX

Given a consumption function:

$$C = C_0 + MPC.Y$$

Numerically speaking

$$C = 1500 + 0.75(Y)$$

Differentiating consumption function w.r.t. Y.

$$C'(Q) = 0 + 0.75(1)$$
$$C'(Q) = 0.75 = MPC$$

Which is marginal propensity to consume.

Now, assume imposition of tax.

$$C^t = 1200 + 0.8(Y_d)$$
$$Y_d = Y - T$$
$$T = 100$$
$$C^t = 1200 + 0.8(Y - T)$$
$$C^t = 1200 + 0.8(Y - 100)$$
$$C^t = 1120 + 0.8(Y)$$
$$MPC^t = \{(C^t)'Q\}$$
$$MPC^t = 0 + 0.8(1) = 0.8$$

0.8 is the marginal propensity to consume in presence of tax

TOPIC 095: RELATIONSHIP BETWEEN MARGINAL-COST AND AVERAGE-COST FUNCTIONS USING QUOTIENT RULE

Given a total Cost function in general form.

$$C = C(Q)$$

Average cost function would be:

$$AC = \frac{C(Q)}{Q} \quad \left[\begin{matrix} Q > 0 \\ AC \neq \infty \end{matrix}\right]$$

Rate of change of AC.

$$\frac{d}{dQ}(AC) = \frac{d}{dQ}\left(\frac{C(Q)}{Q}\right) \qquad \left[\frac{d}{dx}\left[\frac{V(x)}{U(x)}\right]\right]$$

$$= \frac{[C'(Q) \cdot Q - C(Q) \cdot 1]}{Q^2}$$

$$\frac{d}{dQ}\left(\frac{C(Q)}{Q}\right) = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right] \gtreqless 0$$

$$\Rightarrow \frac{d}{dQ}\left[\frac{C(Q)}{Q}\right] \gtreqless 0$$

Let $\frac{d}{dQ}\left[\frac{C(Q)}{Q}\right] = 0$

$$\Rightarrow \frac{1}{Q}\left\{C'(Q) - \frac{C(Q)}{Q}\right\} = 0$$

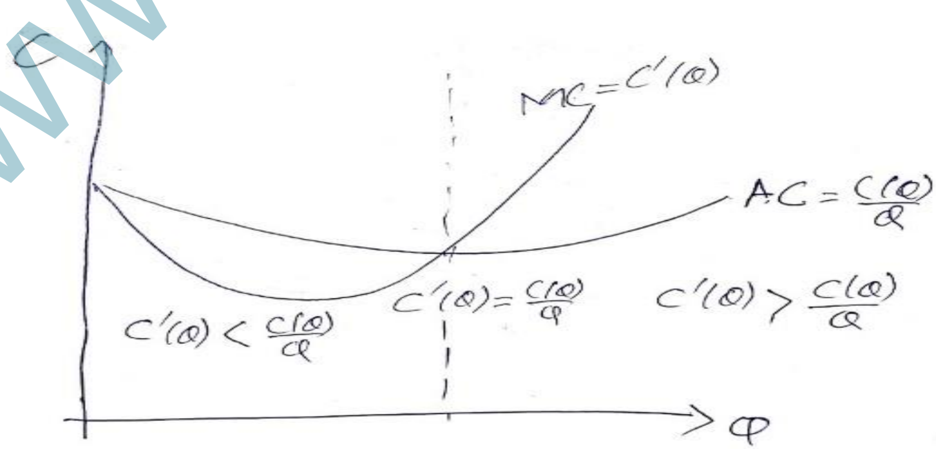
$$C'(Q) - \frac{C(Q)}{Q} = 0$$

$$C'(Q) = \frac{C(Q)}{Q}$$

Similarly $C'(Q) \gtrless \frac{C(Q)}{Q}$

Combined result $C'(Q) \gtreqless \frac{C(Q)}{Q}$

OR $MC \gtreqless AC$



Lesson 21

CHAIN RULE AND INVERSE FUNCTION RULE OF DIFFERENTIATION

TOPIC 096: VARIABLE AND FIXED COST COMPONENTS IN TOTAL COST FUNCTION

In short run, some costs do not change (cost of land, equipment and rent) – Fixed costs (FC)

- However, in long run all costs become variables.
- Other costs vary with output (cost of raw material, components, energy and unskilled labor) – Variable costs (VC).
  Total variable costs: $TVC=(VC).Q$

Total Costs

$$TC = FC + TVC$$
$$TC = FC + (VC).Q$$

Average Costs

$$AC = \frac{TC}{Q}$$
$$AC = \frac{FC + (VC).Q}{Q}$$

$$AC = \frac{FC}{Q} + \frac{(VC).Q}{Q}$$
$$AC = \frac{FC}{Q} + VC$$

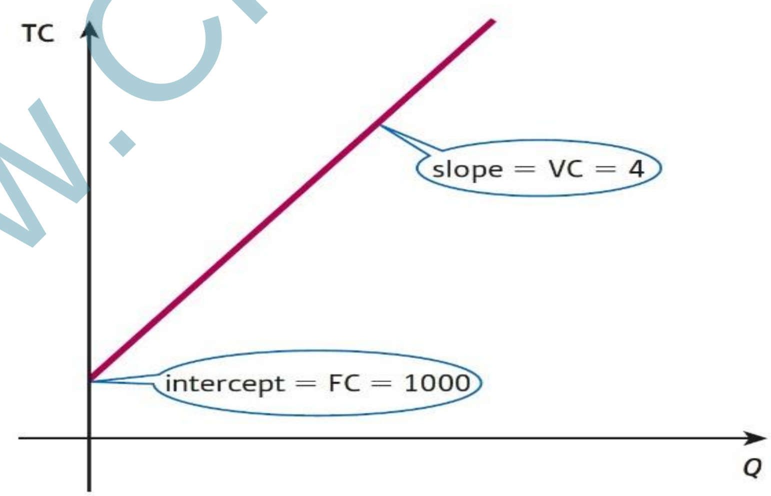
If FC = 1000, VC = 4, then:

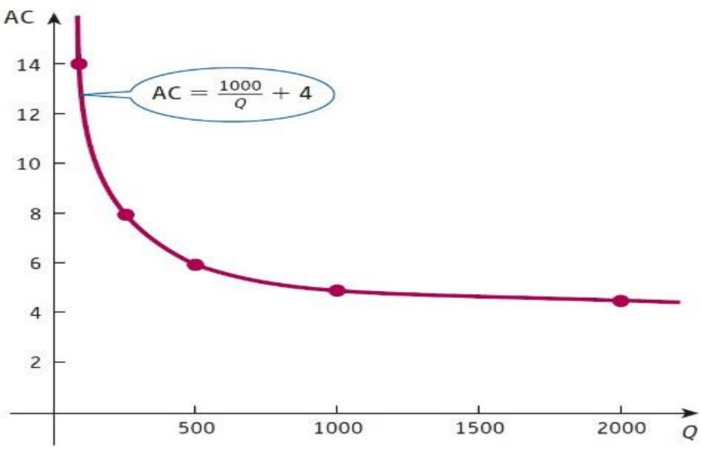
$$TC = 1000 + 4(Q)$$
$$TC = 1000 + 4Q$$

- **Average Cost**

$$AC = \frac{1000 + 4Q}{Q}$$

$$AC = \frac{1000}{Q} + \frac{(4).Q}{Q}$$
$$AC = \frac{1000}{Q} + 4$$





TOPIC 097: OBTAINING MARGINAL COST FUNCTION FROM AVERAGE COST FUNCTION

Given average cost function:

$$AC = Q^2 - 4Q + 174$$

Total cost function

$$AC = \frac{TC}{Q}$$
$$AC \times Q = TC$$
$$TC = AC \times Q$$
$$TC = (Q^2 - 4Q + 174) \times Q$$
$$TC = Q^3 - 4Q^2 + 174Q$$

Marginal cost function

$$MC = \frac{d(TC)}{dQ}$$
$$MC = \frac{d(Q^3 - 4Q^2 + 174Q)}{dQ}$$
$$MC = 3Q^2 - 8Q + 174$$

Long run or Short run?

Consider the total cost function.

$$TC = Q^3 - 4Q^2 + 174Q$$

Since all term involve Q (variable).

All components can vary, which is possible in long run – Long run cost function.

TOPIC 098: MARGINAL COST ANALYSIS

Given total cost function:

$$C = 3Q^2 + 7Q + 12$$

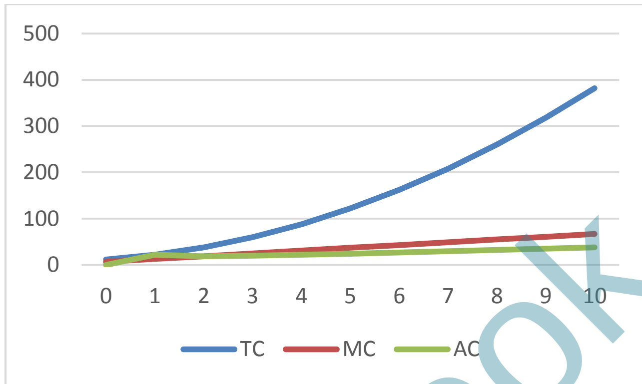
Marginal cost function

$$MC = \frac{d(C)}{dQ}$$
$$MC = \frac{d(3Q^2 + 7Q + 12)}{dQ}$$
$$MC = 6Q + 7$$

$$C = 3Q^2 + 7Q + 12$$

Average cost function

$$AC = \frac{C}{Q}$$
$$AC = \frac{3Q^2 + 7Q + 12}{Q}$$
$$AC = 3Q + 7 + \frac{12}{Q}$$



TOPIC 099: MARGINAL REVENUE ANALYSIS

Given total revenue function:

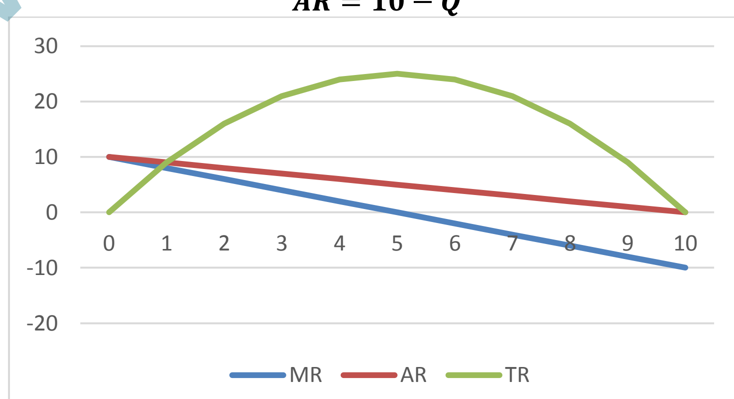
$$R = 10Q - Q^2$$

Marginal revenue function

$$MR = \frac{d(R)}{dQ}$$
$$MR = \frac{d(10Q - Q^2)}{dQ}$$
$$MR = 10 - 2Q$$
$$R = 10Q - Q^2$$

Average revenue function

$$AR = \frac{R}{Q}$$
$$AR = \frac{10Q - Q^2}{Q}$$
$$AR = 10 - Q$$



TOPIC 100: MARGINAL PRODUCT ANALYSIS

Given total product function:

$$Q = aL + bL^2 - cL^3$$

Where, $a, b, c > 0$

Marginal product (of labor) function

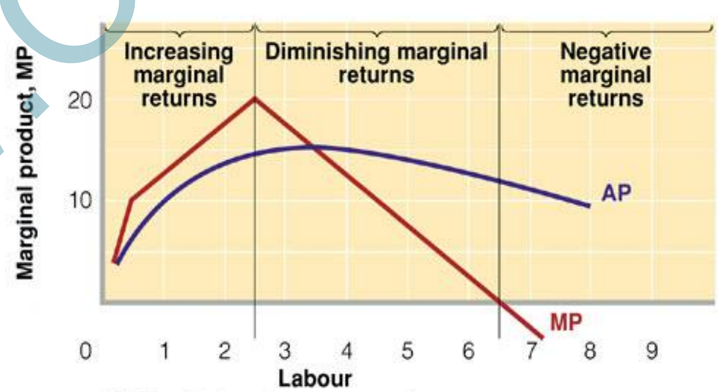
$$MP_L = \frac{d(Q)}{dL}$$
$$MP_L = \frac{d(aL + bL^2 - cL^3)}{dL}$$
$$MP_L = a + 2bL - 3cL^2$$

$$Q = aL + bL^2 - cL^3$$

Average product function

$$AP = \frac{Q}{L}$$
$$AP = \frac{aL + bL^2 - cL^3}{L}$$
$$AP = a + bL - cL^2$$

(a) Total product



(b) Marginal and average product

Lesson 22

USE OF PARTIAL DIFFERENTIATION IN ECONOMICS

TOPIC 101: RULES OF DIFFERENTIATION FUNCTIONS WITH DIFFERENT VARIABLES CHAIN RULE

Possibility of indirect dependence of one variable on other.

$z = f(y)$ where $y = g(x)$

$$z = f\{y(x)\}$$

- Dependence of z on x via y.

$\Rightarrow z = f(x)$

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$\Delta z \xleftarrow{yields} \Delta y \xleftarrow{yields} \Delta x$$

- Hence chain reaction.

Possibility of more than 2 functions.

$z = f(y)$ Where, $y = g(x)$ & $x = h(w)$

$$z = f[y\{x(w)\}]$$

Dependence of z on w via y and x, respectively.

$\Rightarrow z = f(w)$

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} \times \frac{dx}{dw}$$

$$\Delta z \xleftarrow{yields} \Delta y \xleftarrow{yields} \Delta x \xleftarrow{yields} \Delta w$$

3-variables & 4-functions in a chain rule.

Consider following functions

$Z = 3y^2$ and $y = 2x + 5$.

$Z \leftarrow y \leftarrow x$

Differentiate both functions.

$\frac{dZ}{dy} = 6y$ & $\frac{dy}{dx} = 2$

Recalling chain rule.

$\frac{dZ}{dx} = \frac{dZ}{dy} \times \frac{dy}{dx}$

Substituting.

$\frac{dZ}{dx} = 6y\,(2)$

$\frac{dZ}{dx} = 12y$

Returning value of $y = 2x + 5$

$\frac{dZ}{dx} = 12(2x+5)$

$\frac{dZ}{dx} = 24x + 60$ } Answer in terms of x.

TOPIC 102: MARGINAL REVENUE PRODUCT OF LABOR (MRPL) ANALYSIS

Given total revenue function of a firm:

$$R = f(Q)$$

Where, output Q is further a function of labor input (L), or $Q = f(L)$

$$\boldsymbol{R = f\{Q(L)\}}$$
$$\boldsymbol{\Rightarrow R = f(L)}$$

Dependence of $\mathbf{R}$ on $\boldsymbol{L}$ via $\boldsymbol{Q}$.

Situation of chain rule.

$$\frac{dR}{dL} = \frac{dR}{dQ} \times \frac{dQ}{dL}$$
$$\Delta R \xleftarrow{yields} \Delta Q \xleftarrow{yields} \Delta L$$
$$\frac{dR}{dL} = \frac{dR}{dQ} \times \frac{dQ}{dL}$$

Revenue due to labor = Marginal revenue × Product due to labor

$$R'(L) = R'(Q) \times Q'(L)$$
$$MRP_L = MR \times MPP_L$$
$$MRP_L = MR \times MPP_L$$

Marginal revenue product of labor = Marginal revenue × Marginal physical product of labor

TOPIC 103: MARGINAL ANALYSIS OF FISHERY PRODUCTION FUNCTION

Estimated production function for a certain lobster fishery:

F(S,E) **=2.26** $\boldsymbol{S^{0.44}E^{0.48}}$

Where **S = Stock of lobsters**,

E = Effort, and $\boldsymbol{F(S,E)}$ **the catch.**

Marginal Product w.r.t Stock of lobsters.

$$MP_S = \frac{\partial\{F(S,E)\}}{\partial S} = \frac{\partial}{\partial S}(2.26\, S^{0.44}E^{0.48})$$
$$MP_S = 0.9944\frac{E^{0.48}}{S^{0.56}}$$

Marginal product w.r.t effort.

$$MP_E = \frac{\partial\{F(S,E)\}}{\partial E} = \frac{\partial}{\partial E}(2.26\, S^{0.44}E^{0.48})$$
$$MP_E = 1.0848\frac{S^{0.44}}{E^{0.52}}$$

Knowledge of values of ***S*** and ***E*** can give rise to numerical values of $\boldsymbol{MP_S}$ and $\boldsymbol{MP_E}$that will be more interpretable.

TOPIC 104: INVERSE FUNCTION RULE

Given a function:

$$y = f(x)$$

It can be reciprocal can be written as:

$$x = f^{-1}(y)$$

Read as: “x is an inverse function of y”.

$\boldsymbol{f^{-1}(x)}$ is function related to original function $\boldsymbol{f(x)}$ similar to $\boldsymbol{f'(x)}$.

Numerical example

$$y = f(x):\ y = 5x + 25$$

$$\frac{dy}{dx} = 5$$

$$x = f^{-1}(y):\ x = \frac{1}{5}y - 5$$

$$\frac{dx}{dy} = \frac{1}{\left({dy}/{dx}\right)}$$

$$\frac{dx}{dy} = \frac{1}{(5)}$$

$$\frac{dx}{dy} = \frac{1}{5}$$

Numerical example

$$y = f(x):\ y = x^5 + x$$

Finding $\frac{dx}{dy}$ is tricky.

$$x = f^{-1}(y): x = ?$$

$${dy}/{dx} = 5x^4$$

$$\frac{dx}{dy} = \frac{1}{\left({dy}/{dx}\right)}$$

$$\frac{dx}{dy} = \frac{1}{5x^4}$$

Lesson 23

MARKET MODEL ANALYSIS USING PARTIAL DERIVATIVES

TOPIC 105: PARTIAL DIFFERENTIATION: THE CONCEPT

There can be more than one independent variables in a function.
For example, $\boldsymbol{n}$ number of independent variable case.

$$y = f(x_1, x_2, x_3, \dots, x_n)$$

Partial differentiation is suitable in such situation.
Delta $'\boldsymbol{\delta}'$ is used to represent change.

$\left({\delta y}/{\delta x_1}\right)$ is partial derivative of function $\boldsymbol{y}$ w.r.t. independent variable $\boldsymbol{x}$.

Further derivatives will be $\left({\delta y}/{\delta x_2}\right), \left({\delta y}/{\delta x_3}\right), \dots, \left({\delta y}/{\delta x_n}\right)$.

So, the number of partial derivatives shall be equal to the number of independent variables.

Partial Differentiation: Numerical Example

$$y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$$

Partial Derivative w.r.t. x_1

$$\frac{\partial (y)}{\partial x_1} = \frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{\partial}{\partial x_1}(3x_1^2 + x_1 x_2 + 4x_2^2)$$

$$= 3\frac{\partial}{\partial x_1}(x_1^2) + x_2\frac{\partial}{\partial x_1}(x_1) + 4x_2^2\frac{\partial}{\partial x_1}(1)$$

$$= 3(2x_1) + x_2 + 4x_2^2(0)$$

$$= 6x_1 + x_2 + 0$$

$$\frac{\partial y}{\partial x_1} = 6x_1 + x_2$$

$$y_1 = f_1(x_1, x_2) = 6x_1 + x_2$$

Similarly, partial derivative of 'y' w.r.t x_2

$$\frac{\partial}{\partial x_2}(y) = \frac{\partial}{\partial x_2} f(\bar{x}_1, x_2) = \frac{\partial}{\partial x_2}\left(3x_1^2 + x_1x_2 + 4x_2^2\right)$$

$$= 3x_1^2 \frac{\partial}{\partial x_2}(1) + x_1 \cdot \frac{\partial x_2}{\partial x_2} + 4\frac{\partial x_2^2}{\partial x_2}$$

$$= 3x_1^2(0) + x_1 + 4(2x_2)$$

$$= 0 + x_1 + 8x_2$$

$$y_2 = x_1 + 8x_2 = f_2'(x_1, x_2)$$

TOPIC 106: MARGINAL PHYSICAL PRODUCT OF LABOR AND CAPITAL USING PARTIAL DERIVATIVES

Specific Cobb-Douglas production function:

$$Q(K,L) = 96\, K^{0.3} L^{0.7}$$

Where, **K = Stock of capital**,
L = Labor, and **Q(K,L) the output**.

Marginal physical product w.r.t stock of capital.

$$MPP_K = \frac{\partial\{Q(K,L)\}}{\partial K} = \frac{\partial}{\partial K}\left(96\, K^{0.3} L^{0.7}\right)$$

$$= 96\,(0.3 \times K^{-0.7}) L^{0.7}$$

$$MPP_K = 28.8\left(\frac{L}{K}\right)^{0.7}$$

Specific Cobb-Douglas production function:

$Q(K,L) = 96\, K^{0.3} L^{0.7}$ Where,

Where, **K = Stock of capital**,
L = Labor, and **Q(K,L) the output**.

Marginal product w.r.t of labor.

$$MPP_L = \frac{\partial\{Q(K,L)\}}{\partial L} = \frac{\partial}{\partial L}\left(96\, K^{0.3} L^{0.7}\right)$$

$$= 96\,(0.7 \times L^{-0.3}) K^{0.3}$$

$$MPP_L = 67.2\left(\frac{K}{L}\right)^{0.3}$$

Marginal physical products of capital and labor:

$$MPP_K = 28.8\left(\frac{L}{K}\right)^{0.7}$$

$$MPP_L = 67.2\left(\frac{K}{L}\right)^{0.3}$$

Knowledge of values of ***K*** and ***L*** can give rise to numerical values of MPP_K and MPP_L that will be more interpretable.

TOPIC 107: MARGINAL UTILITY FUNCTIONS USING PARTIAL DERIVATIVES

Specific utility function:

$U(x_1,x_2) = (x_1+2)^2(x_2+3)^3$

where x_1 = **1st Good**,

x_2 = **2nd Good**, and $U(x_1,x_2)$ **is the total utility**.

Marginal utility w.r.t 1st good.

$$MU_1 = \frac{\partial\{U(x_1,x_2)\}}{\partial x_1} = \frac{\partial}{\partial x_1}\{(x_1+2)^2(x_2+3)^3\}$$
$$MU_1 = 2(x_1+2)(x_2+3)^3$$

Marginal utility w.r.t 2nd good.

$$MU_2 = \frac{\partial\{U(x_1,x_2)\}}{\partial x_2} = \frac{\partial}{\partial x_2}\{(x_1+2)^2(x_2+3)^3\}$$
$$MU_2 = 3(x_1+2)^2(x_2+3)^2$$

Marginal utilities w.r.t. 1st and 2nd Goods, respectively:

$$MU_1 = 2(x_1+2)(x_2+3)^3$$
$$MU_2 = 3(x_1+2)^2(x_2+3)^2$$

$\{MU_1(x_1,x_2)\}_{(3,3)} = 2160$; knowledge of values of x_1 and x_2 gave rise to numerical value of MU_1 which is more interpretable.

$\{MU_2(x_1,x_2)\}_{(3,3)} = ?$

TOPIC 108: OUTPUT ELASTICITY OF LABOR AND CAPITAL USING PARTIAL DERIVATIVES

General, Cobb-Douglas production function:

$Q(K,L) = A\,K^{\alpha}L^{\beta}$

Where, **K = Stock of capital**,

L = Labor, and **Q(K, L) the output**.

While, α = Output elasticity of capital (ϵ_{QK}) and β = Output elasticity of labor (ϵ_{QL}).

$$\epsilon_{QK} = \left(\frac{\partial Q}{\partial K}\right)\Big/\left(\frac{Q}{K}\right)$$

Numerator

$$\frac{\partial Q}{\partial K} = \frac{\partial\{Q(K,L)\}}{\partial K} = \frac{\partial}{\partial K}(A\,K^{\alpha}L^{\beta})$$
$$= A\alpha K^{\alpha-1}L^{\beta}$$

$$= \alpha A K^{\alpha-1}L^{\beta} = \alpha A K^{\alpha}K^{-1}L^{\beta}$$
$$= \alpha A\frac{K^{\alpha}}{K}L^{\beta}$$
$$\frac{\partial Q}{\partial K} = \alpha\frac{AK^{\alpha}L^{\beta}}{K}$$

As

$$\epsilon_{QK} = \left(\frac{\partial Q}{\partial K}\right)\Big/\left(\frac{Q}{K}\right)$$
$$= \left(\alpha\frac{AK^{\alpha}L^{\beta}}{K}\right)\Big/\left(\frac{AK^{\alpha}L^{\beta}}{K}\right)$$
$$= \left(\alpha\frac{AK^{\alpha}L^{\beta}}{K}\right)\left(\frac{K}{AK^{\alpha}L^{\beta}}\right)$$

$$\epsilon_{QK} = \alpha$$

D.I.Y for ϵ_{QL} =?

TOPIC 109: MONEY MARKET ANALYSIS USING PARTIAL DERIVATIVES

Money supply (M) has two components: cash holdings (C) and bank deposits (D) .

Assume a constant ratio $\left(\frac{C}{D} = c\right)$ Where, $(0 < c < 1)$.

High-powered money (H) is the sum of cash holdings (C) held by the public and the reserves (R) held by the banks.

Bank reserves are a fraction of bank deposits, determined by the reserve ratio. $\left(\frac{R}{D} = r\right)$ Where, $(0 < r < 1)$.

Using given information following tasks can be done:

- Money supply expressed in terms of high-powered money.
- Impact of reserve ratio and money supply
- Impact of cash-deposit ratio and money supply.

Equation of Money Supply:

$M = C + D$ [Narrow Money]

Given a constant ratio b/w C & D

$C/D = c$, where $(0 < c < 1)$

(cash holdings as a ratio of deposits)

Equation of High-powered Money:

$H = C + R$ where R = Bank reserves

$R/D = r$, where $(0 < r < 1)$

(Bank reserves as a ratio of deposits)

Expression of Money Supply in terms of High-powered Money

$$M = f(H)$$

Money Supply:

$M = C + D$

since $C/D = c$

$\Rightarrow C = cD$

$\therefore M = cD + D$

$M = D(1+c)$

High powered Money:

$H = C + R$

since $R/D = r$ & $C/D = c$

$R = rD$ & $C = cD$

$H = cD + rD$

$H = D(c+r)$

$\frac{H}{c+r} = D$

$M = \frac{H}{c+r}(1+c) \Rightarrow \boxed{M = \left(\frac{1+c}{c+r}\right)H = f(H)}$

Impact of Reserve Ratio (r) on Money Supply (M)

Using equation of money supply in terms of H.

$$M = \left(\frac{1+c}{c+r}\right)H$$

Partial Derivative w.r.t. r.

$$\frac{\partial M}{\partial r} = H(1+c)\frac{\partial}{\partial r}\left\{\frac{1}{(c+r)}\right\}$$

$$= H(1+c)\left\{\frac{\partial}{\partial r}(c+r)^{-1}\right\}$$

$$= H(1+c)\left\{(-1)(c+r)^{-2}\frac{\partial}{\partial r}(c+r)\right\}$$

$$= H(1+c)\left\{\frac{-1}{(c+r)^2}\right\}$$

$$\frac{\partial M}{\partial r} = -H\left\{\frac{(1+c)}{(c+r)^2}\right\} < 0 \Rightarrow M \propto \frac{1}{r}$$

$r\uparrow, M\downarrow$ & $r\downarrow, M\uparrow$

Impact of Cash-Deposit ratio (c) on Money Supply:

As $M = \left(\frac{1+c}{c+r}\right) H$

Partial derivative w.r.t c.

$$\frac{\partial M}{\partial c} = H \cdot \frac{\partial}{\partial c}\left(\frac{1+c}{c+r}\right)$$

$$= H\left\{\frac{(c+r)(1) - (1+c)(1)}{(c+r)^2}\right\}$$

Applying Quotient Theorem of differentiation.

$$= H\left\{\frac{c + r - 1 - c}{(c+r)^2}\right\}$$

$$\frac{\partial M}{\partial c} = H\left\{\frac{(r-1)}{(c+r)^2}\right\} \gtrless 0$$

Remarks As $0 < r < 1$

$r > 0$ (+ve)

$r < 1$ (fraction) $\Rightarrow r - 1 < 0$

$(r-1)$ is -ve.

$$\frac{\partial M}{\partial c} = H\left\{\frac{(r-1)}{(c+r)^2}\right\} < 0 \Rightarrow M \propto \frac{1}{c}$$

$c\uparrow, M\downarrow$ & $c\downarrow, M\uparrow$

Lesson 24

SECOND AND HIGHER ORDER DERIVATIVES

TOPIC 110: PARTIAL MARKET MODEL USING PARTIAL DIFFERENTIATION

Money supply (M) has two components: Cash holdings (C) and bank deposits (D) .
Assume a constant ratio $\left(\frac{C}{D} = c\right)$ Where, $(0 < c < 1)$.
High-powered money (H) is the sum of cash holdings (C) held by the public and the reserves (R) held by the banks.

Borrowing the Demand & Supply equations in their Standard Forms.

$Q = a - b.P$ $\quad (a, b, c, d) > 0$

$Q = -c + d.P$ $\quad$ All parameters are +ive.

Equilibrium values of endogenous variables Q^* & P^* are as follows:

$$P^* = \frac{a+c}{b+d} \quad \& \quad Q^* = \frac{ad-bc}{b+d}$$

Reduced forms (in terms of parameters)

Effect of 'a' on P^* via partial derivative.

$$\frac{\partial (P^*)}{\partial a} = \frac{\partial}{\partial a}\left\{\frac{a+c}{b+d}\right\} = \left(\frac{1}{b+d}\right)\frac{\partial}{\partial a}(a+c)$$

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0 \quad \text{—— (1)}$$

Effect of 'b' on P^* via partial derivative.

$$\frac{\partial (P^*)}{\partial b} = \frac{\partial}{\partial b}\left\{\frac{a+c}{b+d}\right\} = (a+c)\frac{\partial}{\partial b}\left\{\frac{1}{b+d}\right\}$$

$$= (a+c)\frac{\partial}{\partial b}(b+d)^{-1} = (a+c)(-1)(b+d)^{-2}(1)$$

$$\frac{\partial P^*}{\partial b} = -\frac{(a+c)}{(b+d)^2} < 0 \quad \text{—— (2)}$$

Effect of 'c' on p^* via partial derivative

$$\frac{\partial(p^*)}{\partial c} = \frac{\partial}{\partial c}\left\{\frac{a+c}{b+d}\right\} = \left(\frac{1}{b+d}\right)\frac{\partial}{\partial c}(a+c)$$

$$\frac{\partial p^*}{\partial c} = \frac{1}{b+d} > 0 \quad \text{——— (3)}$$

Effect of 'd' on p^* via partial derivative

$$\frac{\partial}{\partial d}(p^*) = \frac{\partial}{\partial d}\left\{\frac{a+c}{b+d}\right\} = (a+c)\frac{\partial}{\partial d}\left(\frac{1}{b+d}\right)$$

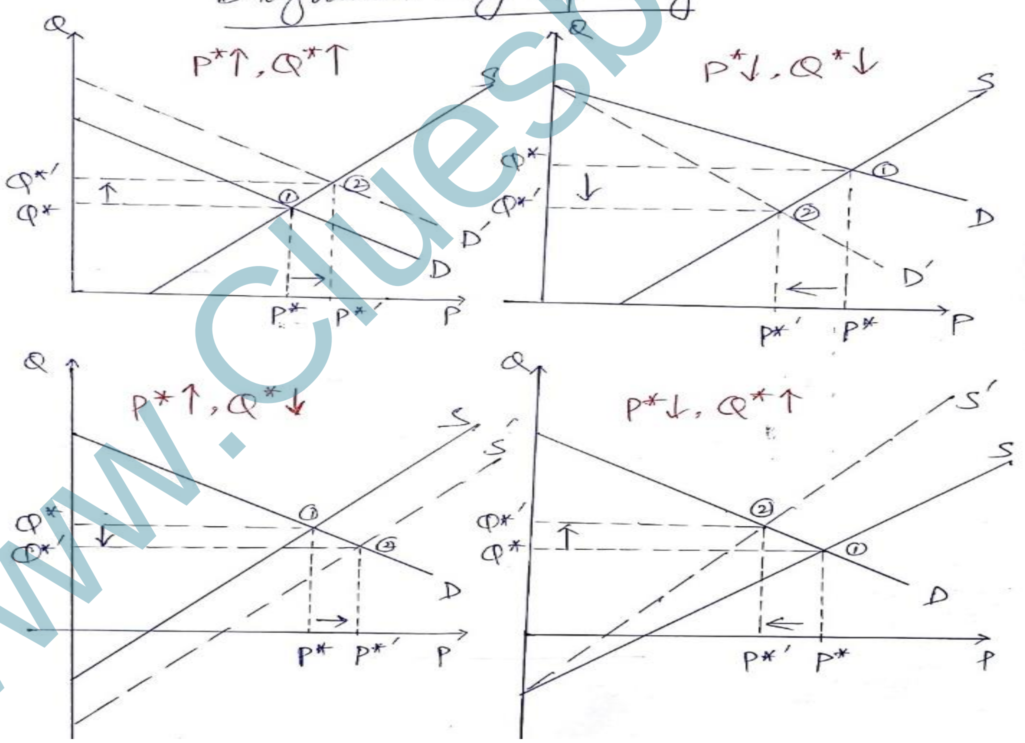
$$= (a+c)\frac{\partial}{\partial d}(b+d)^{-1} = (a+c)(-1)(b+d)^{-2}(1)$$

$$\frac{\partial p^*}{\partial d} = -\frac{(a+c)}{(b+d)^2} < 0 \quad \text{——— (4)}$$

Analysing eq (1), (2), (3) & (4) collectively,

$$\frac{\partial p^*}{\partial a} = \frac{\partial p^*}{\partial c} > 0 \quad \& \quad \frac{\partial p^*}{\partial b} = \frac{\partial p^*}{\partial d} < 0$$

Diagramatically Speaking:



TOPIC 111: NATIONAL-INCOME MODEL USING PARTIAL DIFFERENTIATION

Borrowing Structural Equations of 3-sector Economy

$$Y = C + I_0 + G_0$$

$$C = \alpha + \beta(Y - T) \quad (\alpha > 0;\ 0 < \beta < 1)$$

$$T = \gamma + \delta Y \quad (\gamma > 0;\ 0 < \delta < 1)$$

Reduced form of Y^*:

$$Y^* = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

- Six parameters & exogenous variables are present in the above-mentioned equations. viz. $(\alpha, \beta, \gamma, I_0, G_0, \delta)$
- Consider G_0, γ and δ & their effect on Y^*.

$$\frac{\partial Y^*}{\partial G_0} = \frac{\partial}{\partial G_0}\left\{\frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}\right\}$$

$$= \left(\frac{1}{1 - \beta + \beta\delta}\right)\frac{\partial}{\partial G_0}(\alpha - \beta\gamma + I_0 + G_0)$$

$$= \left(\frac{1}{1 - \beta + \beta\delta}\right) > 0 \quad \left[\because \beta < 1 \Rightarrow (1-\beta) > 0 \ \&\ (1-\beta) + \beta\delta > 0 \Rightarrow \left(\frac{1}{1-\beta+\beta\delta}\right) > 0\right]$$

$\frac{\partial Y^*}{\partial G_0}$ = Government-expenditure Multiplier.

$$\frac{\partial Y^*}{\partial \gamma} = \frac{\partial}{\partial \gamma}\left\{\frac{\alpha - \beta\gamma + I_0 + G_0}{1-\beta+\beta\delta}\right\}$$

$$= \left(\frac{1}{1-\beta+\beta\delta}\right)\frac{\partial}{\partial \gamma}\left\{\alpha - \beta\gamma + I_0 + G_0\right\}.$$

$$\frac{\partial Y^*}{\partial \gamma} = \frac{-\beta}{1-\beta+\beta\delta} < 0 \quad \left[\text{Using the virtue in } \frac{\partial Y^*}{\partial G_0}\right]$$

As γ = non-income tax or autonomous tax, this expression ∂Y^* can be called Non-income tax multiplier.

$$\frac{\partial Y^*}{\partial \delta} = \frac{\partial}{\partial \delta}\left\{\frac{\alpha - \beta\gamma + I_0 + G_0}{1-\beta+\beta\delta}\right\}$$

$$= (\alpha - \beta\gamma + I_0 + G_0)\frac{\partial}{\partial \delta}(1-\beta+\beta\delta)^{-1}$$

$$= (\alpha - \beta\gamma + I_0 + G_0)(-1)(1-\beta+\beta\delta)^{-2}\frac{\partial}{\partial \delta}(1-\beta+\beta\delta)$$

$$= -\frac{(\alpha - \beta\gamma + I_0 + G_0)}{(1-\beta+\beta\delta)^2}\cdot(\beta)$$

$$= \frac{-\beta}{1-\beta+\beta\delta}\cdot\left(\frac{\alpha - \beta\gamma + I_0 + G_0}{1-\beta+\beta\delta}\right)$$

$$\frac{\partial Y^*}{\partial \delta} = \left(\frac{-\beta}{1-\beta+\beta\delta}\right)\cdot Y^* \Rightarrow \text{A fraction of } Y^* \text{ being reduced}$$

↑ to what extent δ (income tax) will reduce Y^*

TOPIC 112: SECOND AND HIGHER ORDER DERIVATIVES

Let, first derivative be $f'(x)$ of a function y = $f(x)$.
Higher-order derivatives can also be calculated.
They are also called higher derivatives.
e.g. Second order derivative $f''(x)$ is twice differentiation of $f(x)$.
Also denoted by $\frac{d^2y}{dx^2}$.

Further, (higher than second) derivatives can be expressed as follows:

$$f'''(x), f^{(4)}(x), \dots, f^{(n)}(x)$$

Or by using notation:

$$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots, \frac{d^ny}{dx^n}$$

Numerical example

Find 1st till fifth derivative of the function:

$$y = f(x) = 4x^4 - x^3 + 17x^2 + 3x - 1$$

1st order derivative.

$$\frac{dy}{dx} = f'(x) = 16x^3 - 3x^2 + 34x + 3$$

2nd order derivative

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = 48x^2 - 6x + 34.$$

3rd order derivative

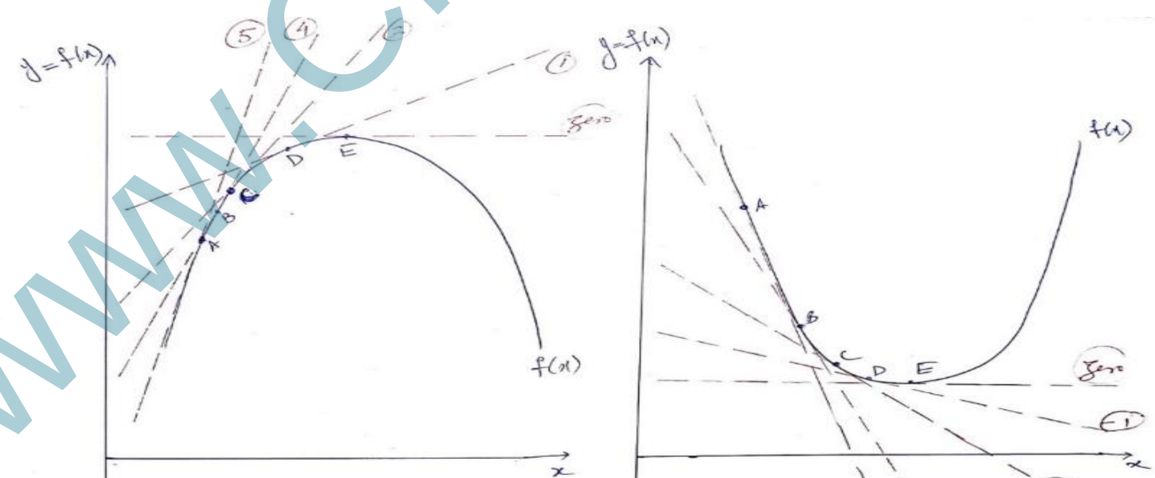
$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = 96x - 6$$

4th order derivative

$$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = f^{(4)}(x) = 96$$

5th order derivative

$$\frac{d}{dx}\left(\frac{d^4y}{dx^4}\right) = \frac{d^5y}{dx^5} = f^{(5)}(x) = 0$$



Rate of change of slope of f(x) is decreasing, $f''(x)<0$

Rate of change of slope of f(x) is increasing, $f''(x)>0$

Rehearsal question

Find first four derivatives of :

$y = g(x) = \frac{x}{1+x}$, where $x \neq -1$

TOPIC 113: ECONOMIC APPLICATIONS OF SECOND DERIVATIVE: PROFIT MAXIMIZATION CONDITION

Firm's Profit Maximization Condition.

Firm's revenue function

$R = f(Q)$

R = revenue & Q = output.

Firm's cost function.

$C = f(Q)$

C = cost of production

Profit function is the difference of these functions.

$\pi(Q) = R f(Q)$

$\pi(Q) = R(Q) - C(Q)$

For maximization/minimization (optimization), first order condition is

$f'(x) = 0$

So, taking first-order derivative w.r.t Q.

$\frac{d\pi}{dQ} = \pi'(Q) = R'(Q) - C'(Q)$ ——— (1)

Following the first-order condition.

$\pi'(Q) = 0$

$0 = R'(Q) - C'(Q)$

$R'(Q) = C'(Q)$ where $Q = Q^*$

↑

Critical value

profit maximizing value of output.

In more suitable terms.

$MR = MC$

For the confirmation of maximum/minimum, second order derivative $f''(x) < 0$ / $f''(x) > 0$

Taking second derivative using eq. ①

$$\frac{d^2\pi}{dQ^2} = \pi''(Q) = R''(Q) - C''(Q)$$

For a maximum

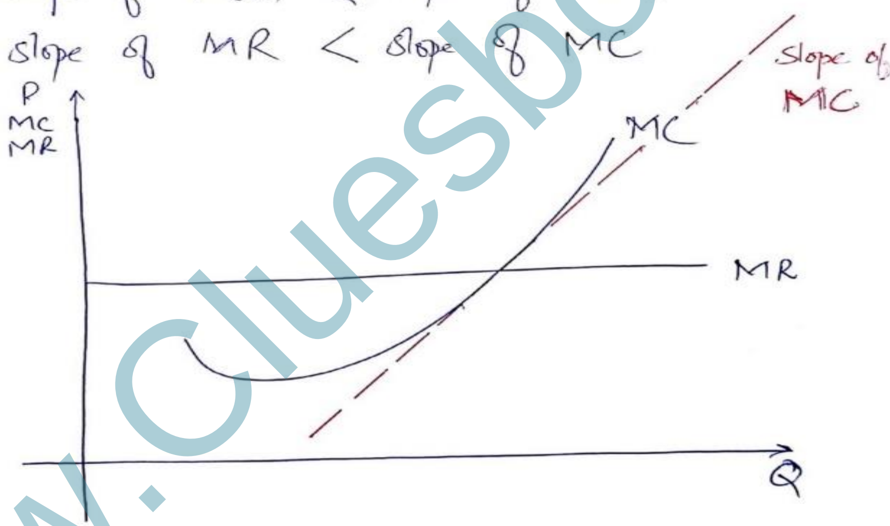
$$R''(Q) - C''(Q) < 0$$

$$R''(Q) < C''(Q)$$

$$\frac{d}{dQ}(R'(Q)) < \frac{d}{dQ}(C'(Q))$$

slope of $R'(Q)$ < slope of $C'(Q)$

slope of MR < slope of MC



TOPIC 114: NUMERICAL EXAMPLE OF PROFIT MAXIMIZATION CONDITION USING SECOND DERIVATIVE

Given $R(Q)$ and $C(Q)$

$$R(Q) = 1200Q - 2Q^2$$
$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$

Forming profit function:

$$\pi(Q) = R(Q) - C(Q)$$
$$= (1200Q - 2Q^2) - (Q^3 - 61.25Q^2 + 1528.5Q + 2000)$$

$$\pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

Applying first order condition to find critical values of Q.

$$\pi'(Q) = -3Q^2 + 118.5Q - 328.5 = 0$$

$$\Rightarrow 3Q^2 - 118.5Q + 328.5 = 0$$

Solving the quadratic equation.

$$Q^* = 3 \quad \text{and} \quad Q^* = 36.5$$

Critical values of Q.

Applying second order condition to choose the Q that leads to maximum.

$$\pi''(Q) = -6Q + 118.5$$

$[\pi''(Q)]_{Q=3} = -6(3) + 118.5$	$[\pi''(Q)]_{Q=36.5} = -6(36.5) + 118.5$
$\pi''(3) = 100.5 > 0$	$\pi''(36.5) = -100.5 < 0$
$f''(x) > 0 \Rightarrow$ minimum	$f''(x) < 0 \Rightarrow$ maximum

$[\pi(Q)]_{Q=3} = D.I.Y$

$[\pi(Q)]_{Q=36.5} = 16,318.44$ units

$\therefore$ Profit maximizing output is

$\underline{Q^* = 36.5}$ &

maximized profit is

$\underline{\pi^* = 16,318.44}$

Lesson 25

PARTIAL DERIVATIVES APPLICATION ON ELASTICITY AND PRODUCTION FUNCTIONS

TOPIC 115: YOUNG'S THEOREM

If $z = f(x_1, x_2, \ldots, x_n)$, then the two second-order cross-partial derivatives z''_{ij} and z''_{ji} are usually equal. That is,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

This implies that the order of differentiation does not matter.

In particular, for the case when $m = 2$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \qquad (i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n)$$

$$Z = f(x, y)$$

$$Z = x^2 + y^2$$

1st order derivative w.r.t x	1st order derivative w.r.t y
$\frac{\partial Z}{\partial x} = 2x + 0$	$\frac{\partial Z}{\partial y} = 0 + 2y$
$\frac{\partial Z}{\partial x} = 2x$	$\frac{\partial Z}{\partial y} = 2y$

2nd order derivative w.r.t y.	2nd order derivative w.r.t x
$\frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}\right) = 0$	$\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right) = 0$
$\frac{\partial}{\partial y}(Z_x) = 0$	$\frac{\partial}{\partial x}(Z_y) = 0$
$Z_{xy} = 0$	$Z_{yx} = 0$

equal

$$\boxed{Z_{xy} = Z_{yx}}$$

TOPIC 116: DEMAND FOR MONEY FUNCTION ANALYSIS USING PARTIAL DERIVATIVES

Demand for Money Function (M) in the US for the period 1929–1952 has been estimated as:

$$M = 0.14Y + 76.03(r-2)^{-0.84} \quad (r > 2)$$

$$M = f(Y, r)$$

Demand for money depends on income (Y) and interest rate (r)

To quantify the impact of income and interest rate on demand for money, we calculate $\partial M/\partial Y$ and $\partial M/\partial r$, respectively.

$$\frac{\partial M}{\partial Y} = \frac{\partial}{\partial Y}\left\{0.14Y + 76.03(r-2)^{-0.84}\right\}$$

$$= \frac{\partial}{\partial Y}\left\{0.14Y\right\} + 76.03\frac{\partial}{\partial Y}\left\{(r-2)^{-0.84}\right\}$$

$$= 0.14(1) + 76.03(0)$$

$$\frac{\partial M}{\partial Y} = 0.14 > 0 \Rightarrow Y\uparrow, M\uparrow$$

With each unit increase in income, the demand for money increase by 0.14 units

$$\frac{\partial M}{\partial r} = \frac{\partial}{\partial r}\left\{0.14Y + 76.03(r-2)^{-0.84}\right\}$$

$$= \frac{\partial}{\partial r}(0.14Y) + 76.03\left\{\frac{\partial}{\partial r}(r-2)^{-0.84}\right\}$$

$$= 0 + 76.03\left\{(-0.84)(r-2)^{-0.84-1}\frac{\partial}{\partial r}(r-2)\right\}$$

$$= 76.03(-0.84)(r-2)^{-1.84}(1)$$

$$\frac{\partial M}{\partial r} = -63.865(r-2)^{-1.84}$$

Let $r = 4$ as $r > 2$

$$\left\{\frac{\partial M}{\partial r}\right\}_{r=4} = -63.865(4-2)^{-1.84}$$

$$\{M'(r)\}_{r=4} = -63.865\ (2)^{-1.84}$$

$$= -63.865\ (0.279)$$

$$M'(4) = -17.839 < 0 \qquad r\uparrow, M\downarrow$$

An increase of 1 unit in interest rate shall decrease demand for money by 17.839 units

TOPIC 117: INCOME ELASTICITY OF DEMAND USING PARTIAL DERIVATIVES

Income elasticity of demand (ϵ_Y) shows the percentage change in demand of a good $(\%\Delta Q)$ w.r.t percentage change in income $(\%\Delta Y)$.

$$(\boldsymbol{\epsilon_Y}) = \frac{\%\Delta \boldsymbol{Q}}{\%\Delta \boldsymbol{Y}}$$

$$= \frac{\Delta Q/_Q}{\Delta Y/_Y} = \frac{\Delta Q}{Q} \times \frac{Y}{\Delta Y}$$

$$= \frac{\Delta Q}{\Delta Y} \times \frac{Y}{Q} = \frac{\Delta Q/_{\Delta Y}}{Q/_Y}$$

$$= \frac{MD_Y}{AD_Y}$$

Where, $\mathbf{MD_Y}$**=** Marginal demand function w.r.t income and $\mathbf{AD_Y}$**=** Average demand function w.r.t income.

Considering an elaborated demand function.

$$Q_1 = a - bP_1 + cP_2 + mY$$

Y = income, P_2 = price of substitute.

Income elasticity of demand ϵ_Y is.

$$\epsilon_Y = \frac{\partial Q_1/\partial Y}{Q_1/Y}$$

Assuming a numerical form of demand function

$$Q_b = 4850 - 5P_b + 1.5P_M + 0.1Y$$

Here $Y = 10{,}000$, $P_b = 200$, $P_M = 100$

$$\epsilon_y = \frac{\partial Q_b/\partial Y}{Q_b/Y} \quad \Big\} \text{ Formula}$$

$$\partial Q_b/\partial Y = \frac{\partial}{\partial Y}(4850 - 5P_b + 1.5P_m + 0.1Y)$$

$$= 0 - 5(0) + 1.5(0) + 0.1(1)$$

$$\frac{\partial Q_b}{\partial Y} = 0.1$$

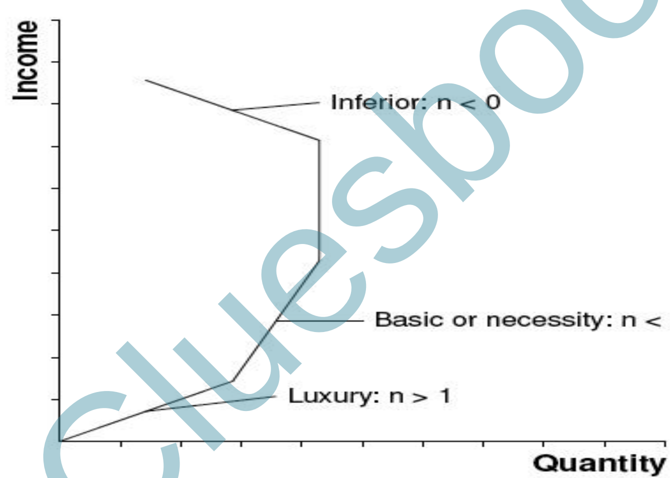
$$Q_b = 4850 - 5(200) + 1.5(100) + 0.1(10{,}000)$$

$$Q_b = 5000$$

$$\epsilon_y = \frac{0.1}{5000/10000} = 0.1 \times \frac{10{,}000}{5{,}000}$$

$$\epsilon_y = 0.2$$

As $\epsilon_y = 0.2 < 1$, the demand curve seems to be less elastic and demand is not likely to be affected much due to changes in income.



TOPIC 118: CROSS PRICE ELASTICITY OF DEMAND USING PARTIAL DERIVATIVES

Cross price elasticity of demand (ϵ_c) shows the percentage change in **demand of one good** ($\%\Delta Q_1$) w.r.t percentage change in **price of other** ($\%\Delta P_2$).

$$(\epsilon_c) = \frac{\%\Delta Q_1}{\%\Delta P_2}$$

$$= \frac{\Delta Q_1/Q_1}{\Delta P_2/P_2} = \frac{\Delta Q_1}{Q_1} \times \frac{P_2}{\Delta P_2}$$

$$= \frac{\Delta Q_1}{\Delta P_2} \times \frac{P_2}{Q_1} = \frac{\Delta Q_1/\Delta P_2}{Q_1/P_2}$$

$$= \frac{MD_{(1,2)}}{AD_{(1,2)}}$$

Where, $\mathbf{MD_{(1,2)}}$= Marginal demand function of Good-1 w.r.t price of Good-2. $\mathbf{AD_{(1,2)}}$= Average demand function of Good-1 w.r.t price of Good-2.

Consider an elaborated demand function.

$$Q_1 = a - bP_1 + cP_2 + mY$$

Y= income, P_2 = price of substitute.

Cross price elasticity of demand

$$\epsilon_c = \frac{\partial Q_1/\partial P_2}{Q_1/P_2}$$

Specifically, a numerical form of such demand function:

$$Q_b = 4850 - 5P_b + 1.5P_M + 0.1Y$$

$Y = 10{,}000$, $P_b = 200$, $P_M = 100$

Partial Derivative w.r.t P_M

$$\frac{\partial Q_b}{\partial P_M} = \frac{\partial}{\partial P_M}(4850 - 5P_b + 1.5P_M + 0.1Y)$$

$$\frac{\partial Q_b}{\partial P_M} = 1.5$$

$$Q_b = 4850 - 5(200) + 1.5(100) + 0.1(10{,}000)$$

$$Q_b = 5000$$

$$\epsilon_c = \frac{\partial Q_b/\partial P_M}{Q_b/P_M} = \frac{1.5}{5000/100}$$

$\epsilon_c = 0.03 > 0$, therefore good 'M' is a substitute to good 'b'.

If $\epsilon_c < 0$, then goods under-discussion are 'complements'

If $\epsilon_c = 0$, " " " are un-related.

TOPIC 119: PARTIAL DERIVATIVES APPLICATION ON HERRING PRODUCTION FUNCTION

A research by Henderson and Tugwell modelled the herring catch:

$$Y(K,S) = 0.06157\,K^{1.356}\,S^{0.562}$$

K = Catching Effort

S = Herring Stock.

One can find the marginal effect of catching effort and Herring stock using partial derivatives:

$$\frac{\partial Y}{\partial K} = \frac{\partial}{\partial K}\left\{0.06157\,K^{1.356}\,S^{0.562}\right\}$$

$$= 0.06157\,S^{0.562}\left(\frac{\partial}{\partial K}K^{1.356}\right)$$

$$= 0.06157\,S^{0.562} \cdot 1.356\,K^{1.356-1} \cdot \frac{\partial(K')}{\partial K}$$

$$= (0.06157)(1.356)\,S^{0.562} \cdot K^{0.356}$$

$$\frac{\partial Y}{\partial K} = 0.0834 \cdot S^{0.562} \cdot K^{0.356} = Y_K$$

Plugging-in the values of S and K, one can get the numerical responses of Y to additional K.

Similarly,

$$\frac{\partial Y}{\partial S} = \frac{\partial}{\partial S}\{0.06157 K^{1.356} S^{0.562}\}$$

$$= 0.06157 K^{1.356} \frac{\partial}{\partial S}(S^{0.562})$$

$$= 0.06157 K^{1.356} (0.562) S^{-0.438} \frac{\partial (S^1)}{\partial S}$$

$$= (0.06157)(0.562) K^{1.356} S^{-0.438}$$

$$\frac{\partial Y}{\partial S} = 0.0346 K^{1.356} S^{-0.438} = Y_S$$

Knowledge of values of K and S, gives us numerical and more easy-to-interpret value of marginal Herring catch w.r.t Herring stock.

Introducing a change both in K and S i.e. doubling both inputs.

$$Y(K,S) = 0.06157 K^{1.356} S^{0.562}$$

$$Y^*(2K,2S) = 0.06157 (2K)^{1.356} . (2S)^{0.562}$$

$$= 0.06157 . 2^{1.356} . K^{1.356} . 2^{0.562} . S^{0.562}$$

$$= 2^{1.356} . 2^{0.562} . 0.06157 . K^{1.356} . S^{0.562}$$

$$= 2^{1.356+0.562} . (0.06157 K^{1.356} S^{0.562})$$

$$= 2^{1.918} . (Y) = 4(Y)$$

[Output is quadrupled if inputs are doubled

i.e <u>I.R.S</u>

TOPIC 120: PARTIAL DERIVATIVES APPLICATION ON THREE INPUT PRODUCTION FUNCTION

Consider a three input production function with product of their logs.

$$F(K, L, M) = (\ln K)(\ln L)(\ln M)$$

logarithmically differentiating w.r.t K

$$\frac{\partial F(K,L,M)}{\partial \ln K} = \frac{\partial}{\partial \ln K}\{(\ln K)(\ln L)(\ln M)\}$$

$$= (\ln L)(\ln M)\frac{\partial}{\partial \ln K}\{(\ln K)\}$$

$$= (\ln L)(\ln M)\left(\frac{\partial \ln K}{\partial \ln K}\right)$$

$$F_K = (\ln L)(\ln M)$$

logarithmically differentiating w.r.t L

$$\frac{\partial F(K,L,M)}{\partial \ln L} = \frac{\partial}{\partial \ln L}\{(\ln K)(\ln L)(\ln M)\}$$

$$= (\ln K)(\ln M)\left\{\frac{\partial (\ln L)}{\partial \ln L}\right\}$$

$$F_L = (\ln K)(\ln M)$$

logarithmically differentiating w.r.t M

$$\frac{\partial F(K,L,M)}{\partial \ln M} = \frac{\partial}{\partial \ln M}\{(\ln K)(\ln L)(\ln M)\}$$

$$= (\ln L)(\ln K)\left\{\frac{\partial (\ln M)}{\partial \ln M}\right\}$$

$$F_M = (\ln L)(\ln K)$$

CROSS PARTIAL DERIVATIVES.

logarithmically differentiating F_L w.r.t K.

$$\frac{\partial (F_L)}{\partial \ln K} = \frac{\partial}{\partial \ln K}\{(\ln K)(\ln M)\}$$

$$\frac{\partial}{\partial \ln K}(F_L) = (\ln M)\frac{\partial(\ln K)}{\partial \ln K}$$

$$F_{LK} = \ln M$$

logarithmically differentiating F_K w.r.t L.

$$\frac{\partial (F_K)}{\partial \ln L} = \frac{\partial}{\partial \ln L}\{(\ln L)(\ln M)\}$$

EQUAL

$$= (\ln M)\frac{\partial(\ln L)}{\partial \ln L}$$

$$F_{KL} = \ln M \Rightarrow \boxed{F_{LK} = F_{KL}}$$ Validation of Young's Theorem

$$\boxed{F_{KM} = F_{MK} \quad \& \quad F_{LM} = F_{ML}}$$ D.I.Y

Lesson 26

PARTIAL DERIVATIVES APPLICATION ON CONSUMER AND PRODUCER THEORIES

TOPIC 121: ENVELOPE THEOREM

A mathematical theorem based on calculus.
Multiple applications in economics including producer theory.
Here, consumer theory is being subjected.

Objective function
$U = U(x_1, x_2)$
Constraint function
$M = P_1x_1 + P_2x_2$

Lagrangian function
$\boldsymbol{L = U(x_1, x_2) + \lambda(M - P_1x_1 - P_2x_2)}$
Optimized utility function:
$\boldsymbol{V = U(x_1^*, x_2^*)}$
Where, $\boldsymbol{x_1^* = f(p_1,\ p_2, M), x_2^* = f(p_1,\ p_2, M)}$ are the Marshallian demand functions.
Detailed optimized utility function: $\boldsymbol{V(p_1,\ p_2, M) = U\{x_1^*(p_1,\ p_2, M), x_2^*(p_1,\ p_2, M)\}}$

x_1^* & x_2^* are variables and $p_1,\ p_2$ & M are parameters.
Variables in $U(x_1, x_2)$ is now parametrized by $p_1,\ p_2$ & M.
If marginal utility of money is to be found, $\frac{dU}{dM}$ (a total derivative) needs to be found at optimum points.

$$\left.\frac{dU}{dM}\right|_{(x_1^*,\, x_2^*,\, \lambda^*)} = \left(\frac{dU}{dx_1}\frac{dx_1}{dM} + \frac{dU}{dx_2}\frac{dx_2}{dM}\right)\Bigg|_{(x_1^*,\, x_2^*,\, \lambda^*)}$$

Which can be a tedious calculation.

$$\left(\frac{dU}{dx_1}\frac{dx_1}{dM} + \frac{dU}{dx_2}\frac{dx_2}{dM}\right)$$

However, Envelope theorem suggests that a partial derivative $\frac{\partial L}{\partial M}$ can be found instead.
Therefore;

$$\frac{\partial L}{\partial M} = \left.\frac{dU}{dM}\right|_{(x_1^*,\, x_2^*,\, \lambda^*)}$$

Envelope Theorem: Partial derivative of Lagrangian w.r.t a given parameter of variables equals the total derivative of objective function evaluated at its variables' optimum points.
Numerical Exercise: Choose a utility function and a budget constraint and evaluate.

TOPIC 122: ROY'S IDENTITY

Attributed to for French economist René Roy.
One of the two ways to calculate Marshallian demand functions.
(Other being constrained optimization of utility function & budget constraint).i.e.

Objective function

$$\boldsymbol{U = U(x_1, x_2)}$$

Constraint function

$$\boldsymbol{M = P_1x_1 + P_2x_2}$$

Uses ratio of partial derivatives of indirect utility function with respect to price of good (under consideration) and income, respectively.

$$x_i^* \equiv -\frac{\frac{\partial V(p,M)}{\partial P_i}}{\frac{\partial V(p,M)}{\partial M}}$$

For 2 Goods case:

$$x_1^* \equiv -\frac{\frac{\partial V(P_1,P_2,M)}{\partial P_1}}{\frac{\partial V(P_1,P_2,M)}{\partial M}} \text{ and } x_2^* \equiv -\frac{\frac{\partial V(P_1,P_2,M)}{\partial P_2}}{\frac{\partial V(P_1,P_2,M)}{\partial M}}$$

$\frac{\partial V(P_1,P_2,M)}{\partial P_1}$, $\frac{\partial V(P_1,P_2,M)}{\partial P_2}$ & $\frac{\partial V(P_1,P_2,M)}{\partial M}$ are needed.

e.g.:

$$V(P_1,P_2,M) = \sqrt{\frac{(4P_1+P_2)}{P_1P_2}M}$$

$$\frac{\partial V(P_1,P_2,M)}{\partial P_1} = \frac{-\frac{M}{P_1^2}}{2\sqrt{\frac{(4P_1+P_2)}{P_1P_2}M}} \quad \& \quad \frac{\partial V(P_1,P_2,M)}{\partial M} = \frac{\frac{(4P_1+P_2)}{P_1P_2}}{2\sqrt{\frac{(4P_1+P_2)}{P_1P_2}M}}$$

$$x_1^* \equiv -\frac{\frac{\partial V(P_1,P_2,M)}{\partial P_1}}{\frac{\partial V(P_1,P_2,M)}{\partial M}}$$

$$= -\frac{\frac{-\frac{M}{P_1^2}}{2\sqrt{\frac{(4P_1+P_2)}{P_1P_2}M}}}{\frac{\frac{(4P_1+P_2)}{P_1P_2}}{2\sqrt{\frac{(4P_1+P_2)}{P_1P_2}M}}}$$

$$= \frac{\frac{M}{P_1}}{\frac{(4P_1+P_2)}{P_2}} = \frac{M}{P_1}\frac{P_2}{(4P_1+P_2)}$$

$$x_1^*(P_1,P_2,M) = \frac{M}{(4P_1+P_2)}\frac{P_2}{P_1}$$

D.I.Y, similarly for $x_2^*(P_1,P_2,M) = \frac{4M}{(4P_1+P_2)}\frac{P_1}{P_2}$

TOPIC 123: HOTELLING'S LEMMA

Attributed to Harold Hotelling.
Mathematical result used to related supply of a good with producer's profit.
The change in profits from a change in price is equal to the quantity produced.

$$y(p) = \frac{\partial \pi(p)}{\partial p}$$

Specifically speaking for a maximized profit function:

$$\pi^* = F(K^*, L^*)$$

$$\frac{\partial \pi^*}{\partial p} = Q^*$$

$$\frac{\partial \pi^*}{\partial r} = -K^*$$

$\frac{\partial \pi^*}{\partial w} = -L^*$

Numerically speaking.
$\pi = pQ - wK$ Assume($\bar{L}$)
π: Profit
p: Price of output
Q: Quantity of output produced
w: Price of capital input
K: Quantity of capital input employed
Let $p = 4$, $Q = 20$, $w = 2$, $K = 10$.
$\pi_{Before} = (4)(20) - (2)(10) = 60$

After change ($p \uparrow$)
$p = 6$, $Q = 20$, $w = 2$, $K = 10$.

$$\pi_{After} = (6)(20) - (2)(10) = 100$$

$\Delta\pi = \pi_{After} - \pi_{Before}$

$\Delta\pi = 100 - 60 = 40$

$\Delta P = P_{After} - P_{Before}$

$\Delta P = 4 - 6 = 2$

$\frac{\partial \pi^*}{\partial p} = \frac{\Delta \pi^*}{\Delta p} = \frac{40}{2} = 20 = Q$

Δ in profits due to Δ in price is 20 equals output produced.

<u>TOPIC 124: SHEPHARD LEMMA</u>
Attributed to Ronald Shephard
Mathematical result used in consumer & producer theory.
Demand for a particular good i for a given level of utility u and prices p, equals the derivative of the expenditure function with respect to the price of the relevant good:

Consumers' point of view

$h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial P_i}$

$h_i(\mathbf{p}, u)$ is the Hicksian demand function for good i.
$e(\mathbf{p}, u)$ is the expenditure function.

Producers' point of view

$x_i(\mathbf{w}, y) = \frac{\partial c(\mathbf{w}, y)}{\partial w_i}$

$x_i(\mathbf{w}, y)$ is the conditional factor demand function for input i.
$c(\mathbf{w}, y)$ is the cost function.

Consumer Theory

$h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial P_i}$

If maximized expenditure function:

$e(P_x, P_y, \bar{U}) = 2P_x^{1/2} P_y^{1/2} \bar{U}^{\frac{1}{2}}$

Hicksian demand function for x and y are:

$$\frac{\partial e(P_x, P_y, \bar{U})}{\partial P_x} = x^h = \frac{P_y^{\frac{1}{2}} \bar{U}^{\frac{1}{2}}}{P_x^{1/2}}$$

$$\frac{\partial e(P_x, P_y, \bar{U})}{\partial P_y} = y^h = \frac{P_x^{\frac{1}{2}} \bar{U}^{\frac{1}{2}}}{P_y^{1/2}}$$

Similarly, producer theory also uses Shephard's Lemma.
D.I.Y by using a cost objective function and an output constraint.

Lesson 27

USE OF DIFFERENTIALS IN ECONOMICS

TOPIC 125: DIFFERENTIALS VERSUS DERIVATIVES

Derivative $\left(\frac{dy}{dx}\right)$ of $y = f(x)$: a single entity.
Ratio of two quantities, dy and dx – differentials.
Derivative as the ratio of differential of function by the differential of variable.
If $\frac{dy}{dx} = f'(x)$ is the derivative of $f(x)$, then rearranging:

$$\boldsymbol{dy = \{f'(x)\}\, dx}$$

Differential of function.

Differentiation ⇒ Differentials (dy).
Differentiation w.r.t. x ⇒ Derivatives $\left(\frac{dy}{dx}\right)$.

Rules of Differentials

- **Sum-Difference Rule(s)**
  $\boldsymbol{d\{g(x) \pm h(x)\} = d\{g(x)\} \pm d\{h(x)\}}$
- **Product Rule**
  $\boldsymbol{d\{g(x) \cdot h(x)\} = d\{g(x)\} \cdot h(x) + d\{h(x)\} \cdot g(x)}$
- **Quotient Rule**
  $\boldsymbol{d\left\{\frac{g(x)}{h(x)}\right\} = \frac{[d\{g(x)\}\cdot h(x) - d\{h(x)\}\cdot g(x)]}{\{g(x)\}^2}}$
  Where $\boldsymbol{g(x) \neq 0}$

Examples

$$\boldsymbol{y = f(x) = Ax^a + B}$$
$$\boldsymbol{dy = d(Ax^a + B)}$$

Since, $\boldsymbol{dy = \{f'(x)\} \cdot dx}$

$$\boldsymbol{dy = \left\{Aax^{a-1}\right\} \cdot dx}$$
$$\boldsymbol{y = f(x) = 3x^2 + 10}$$
$$\boldsymbol{dy = d(3x^2 + 10)}$$

Since, $\boldsymbol{dy = \{f'(x)\} \cdot dx}$

$$\boldsymbol{dy = \left\{3 \cdot 2 \cdot x^{2-1}\right\} \cdot dx}$$
$$\boldsymbol{dy = (6x) \cdot dx}$$

TOPIC 126: POINT ELASTICITY USING DIFFERENTIALS

Assume a demand function.

$$Q = f(P)$$

$$\varepsilon_d = \epsilon = \frac{\Delta Q/Q}{\Delta P/P} \cong \frac{dQ/Q}{dP/P}$$ [Replacing change 'Δ' with differential 'd']

The formula for point elasticity.

OR

$$\epsilon = \frac{dQ/dP}{Q/P}$$

- Numerator shows the ratio of two differentials. dQ & dP.
- Another connotation would be:

$$\epsilon = \frac{dQ/dP}{Q/P} = \frac{\text{Marginal demand function}}{\text{Average demand function}}$$

Elasticity of Supply

$$\epsilon_s = \frac{dQ_s/dP}{Q_s/P} = \frac{\text{Marginal supply function}}{\text{Average supply function}}$$

Point elasticity of supply also depends on the ratio of two differentials dQ_s & dP.

Numerically speaking $Q_s = P^2 + 7P$, $P = 2$.

$$\frac{dQ_s}{dP} = 2P + 7$$
$$= 2(2) + 7$$
$$dQ_s/dP = 11$$

$$Q_s/P = \frac{P^2 + 7P}{P}$$
$$= \frac{2^2 + 7(2)}{2}$$

$$Q_s/P = \frac{4 + 14}{2} = 9$$

$$\epsilon_s = \frac{dQ_s/dP}{Q_s/P} = \frac{11}{9} > 1.$$

Since $\epsilon_s > 1$, the supply curve is elastic.

Numerical example:

$$Q = 100 - 2P$$

Resorting to the formula of point elasticity.

$$\epsilon = \frac{dQ/dP}{Q/P}$$

$$\frac{dQ}{dP} = \frac{d}{dP}(100 - 2P) = -2.$$

For $P = 25$, $Q = 100 - 2(25) = 50$

$P = 30$, $Q = 100 - 2(30) = 40$

$$\epsilon = \frac{-2}{50/25} = -1$$

Unitary elastic demand curve

$$\epsilon = \frac{-2}{40/30} = -1.5$$

More elastic demand curve.

TOPIC 127: ELASTICITY OF RECTANGULAR HYPERBOLIC DEMAND CURVE

Assume a demand function

$$Q = k/P^n$$

$$k > 0, \quad n > 0$$

$$\epsilon_d = \frac{dQ/dP}{Q/P}$$

$$\frac{dQ}{dP} = \frac{d}{dP}\left(\frac{k}{P^n}\right)$$

$$= k\frac{d}{dP}(P^{-n})$$

$$= k(-n)P^{-n-1}\frac{d}{dP}(P^1)$$

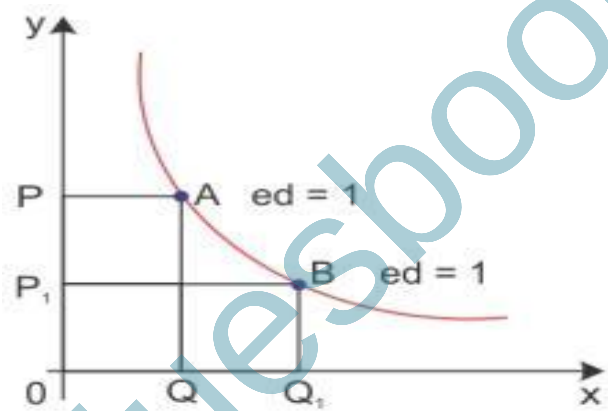
$$\frac{dQ}{dP} = \frac{-nk}{P^{n+1}}$$

$$E_d = \frac{\frac{-nk}{P^{n+1}}}{\frac{k/P^n}{P}} = \frac{-nk}{P^{n+1}} \times \frac{P^n}{k} \cdot P$$

$$= \frac{-nk}{k} \cdot \frac{P^{n+1}}{P^{n+1}}$$

$\boxed{E_d = -n}$ a constant value, free of variable 'P'.

If $n = 1$, then the shape of demand curve shall be a Rectangular Hyperbola



TOPIC 128: INCOME AND PRICE ELASTICITY OF DEMAND USING DIFFERENTIALS

Assume a demand function, in terms of price and income.

$$Q = 100 - 2P + 0.02Y \quad , \quad P = 20, \; Y = 5000$$

Price elasticity of demand

$$E_p = \frac{dQ/dP}{Q/P} = \frac{\partial Q/\partial P}{Q/P}$$

$$\frac{\partial Q}{\partial P} = \frac{\partial}{\partial P}(100 - 2P + 0.02Y)$$

$$\boxed{\frac{\partial Q}{\partial P} = -2}$$

$Q = 100 - 2(20) + 0.02(5000)$

$\boxed{Q = 160}$

$\epsilon_p = \frac{-2}{160/20} = -2 \times \frac{20}{160}$

$\boxed{\epsilon_p = -1/4}$ Elasticity of demand w.r.t Price is low – Less price-elastic

$\boxed{\epsilon_p = -0.25}$

Income elasticity of demand

$\epsilon_Y = \frac{\partial Q/\partial Y}{Q/Y}$

$\frac{\partial Q}{\partial Y} = \frac{\partial}{\partial Y}(100 - 2P + 0.02Y)$

$\frac{\partial Q}{\partial Y} = 0.02$

$\epsilon_Y = \frac{0.02}{160/5000} = 0.02 \times \frac{5000}{160}$

$\boxed{\epsilon_Y = 0.625}$ Elasticity of demand w.r.t income is also low – less income-elastic.

Interpretative Comparison: $\epsilon_p = -0.25$

$|\epsilon_p| = 0.25$

$\epsilon_Y = 0.625$

$\epsilon_Y > |\epsilon_p|$

Therefore demand curve is relative terms is more elastic w.r.t income as compared to price

TOPIC 129: INCOME ELASTICITY OF CONSUMPTION USING DIFFERENTIALS

Consider the standard form of Consumption function

$$C = a + bY \quad \left\{ \begin{array}{l} a > 0 \\ 0 < b < 1 \end{array} \right\}$$

Elasticity of consumption w.r.t income ϵ_{CY}

$$\epsilon_{CY} = \frac{dC/dY}{C/Y}$$

$$\frac{d(C)}{dY} = \frac{d}{dY}(a+bY)$$

$$\frac{dC}{dY} = b$$

$$= \frac{b}{\left(\frac{a+bY}{Y}\right)}$$

$$\epsilon_{CY} = \frac{bY}{a+bY}$$

$\epsilon_{CY} > 0$ [Since $a > 0, b > 0$]

$\epsilon_{CY} < 1$ [Since $bY < a + bY$]

$$\rightarrow \frac{\% \Delta C}{\% \Delta Y} < 1$$

$\% \Delta C < \% \Delta Y$ Keynes Law of consumption.

A specific form of consumption function

$$C = 20 + 0.8Y$$

$$\frac{dC}{dY} = \frac{d}{dY}(20 + 0.8Y)$$

$$\frac{dC}{dY} = 0.8$$

Let $Y = 100$

$$C = 20 + 0.8(100)$$

$$C = 20 + 80$$

$$C = 100$$

$$\epsilon_{CY} = \frac{dC/dY}{C/Y}$$

$$\epsilon_{CY} = \frac{0.8}{(100/100)}$$

$$\boxed{\epsilon_{CY} = 0.8}$$

$\epsilon_{CY} > 0$ +ve relationship b/w C and Y.

$\epsilon_{CY} < 1$ less than proportional increase in C.

TOPIC 130: INCOME AND PRICE ELASTICITY OF IMPORT FUNCTION USING DIFFERENTIALS

Consider the import function

$$M = -17.5095 + 3.4765Y - 1.6418P_{ID}$$

M = imports

Y = Income (GDP)

P_{ID} = Ratio of import prices to domestic prices

For $Y = 1000$ & $P_{ID} = 0.7$

$$M = -17.5095 + 3.4765(1000) - 1.6418(0.7)$$

$$M = 3,457.8$$

$$\epsilon_{MY} = \frac{\partial M / \partial Y}{M/Y}$$

$$\frac{\partial M}{\partial Y} = \frac{\partial}{\partial Y}(-17.5095 + 3.4765\,Y - 1.6418\,P_{ID})$$

$$\frac{\partial M}{\partial Y} = 3.4765$$

$$\epsilon_{MY} = \frac{3.4765}{\frac{3457.8}{1000}} = 3.4765 \times \frac{1000}{3457.8}$$

$\epsilon_{MY} = 1.005 > 1$ More elastic imports w.r.t GDP.

$$\epsilon_{MP} = \frac{\partial M / \partial P_{ID}}{M/P_{ID}}$$

$$\frac{\partial M}{\partial P_{ID}} = \frac{\partial}{\partial P_{ID}}(-17.5095 + 3.4765\,Y - 1.6418\,P_{ID})$$

$$\frac{\partial M}{\partial P_{ID}} = -1.6418$$

$$\epsilon_{MP} = \frac{-1.6418}{\frac{3457.8}{0.7}} = -1.6418 \times \frac{0.7}{3457.8}$$

$\epsilon_{MP} = -0.00033 < 0$ quite low elasticity w.r.t P_{ID}.

$|\epsilon_{MP}| = 0.00033 < 1$

Comparative Interpretation:

$$\epsilon_{MY} > |\epsilon_{MP}|$$

$$1.005 > |-0.00033|$$

TOPIC 131: OUTPUT ELASTICITY OF COST

Considering a cost function

$$C = 0.0004Q^2 + 8Q + 64000$$

$$\frac{dC}{dQ} = 0.0008Q + 8$$

At $Q = 1000$

$$\left[\frac{dC}{dQ}\right]_{Q=1000} = 0.0008(1000) + 8$$

$$\left.\frac{dC}{dQ}\right|_{Q=1000} = 8.8$$

$$\frac{C}{(Q=1000)} = 0.0004Q^2 + 8Q + 64000$$

$$= 0.0004(1000)^2 + 8(1000) + 64000$$

$$= 72400$$

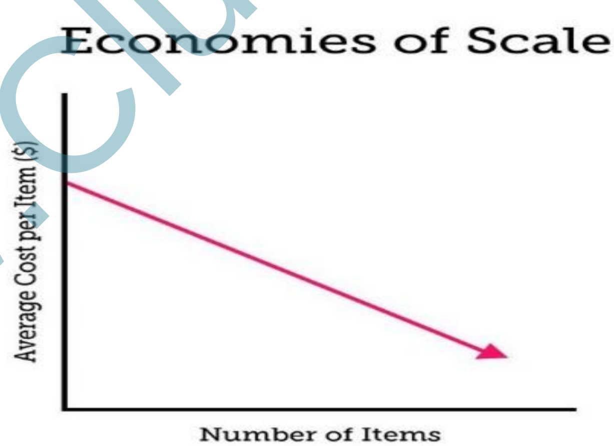
Output elasticity of Cost

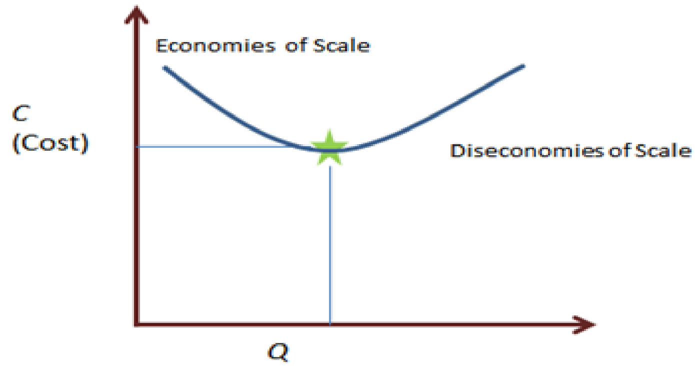
$$\epsilon_{CQ} = \frac{dC/dQ}{C/Q} = \frac{8.8}{\frac{72400}{1000}} = \frac{8.8}{72.4}$$

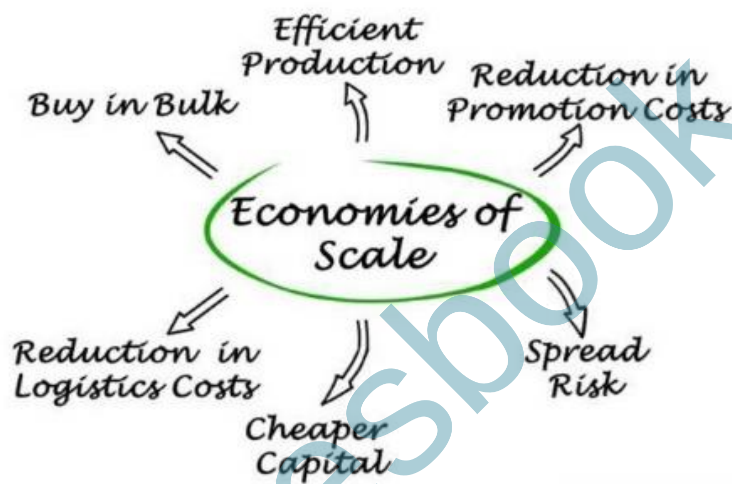
$$\boxed{\epsilon_{CQ} = 0.12}$$

$\epsilon_{CQ} > 0$ [+ve relationship b/w output and cost]

$\epsilon_{CQ} < 1$ [$\% \Delta C < \% \Delta Q \Rightarrow$ Economies of Scale exist]



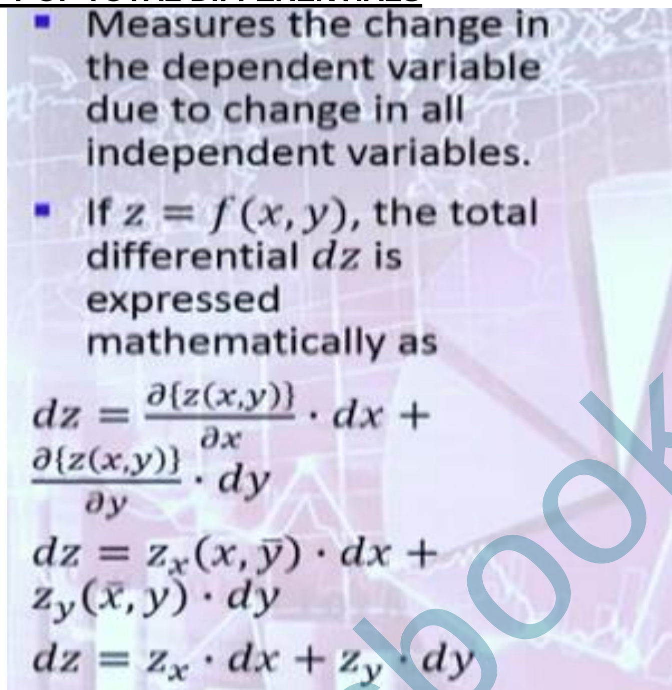

Economies of Scale
C
(Cost)
Diseconomies of Scale
Q

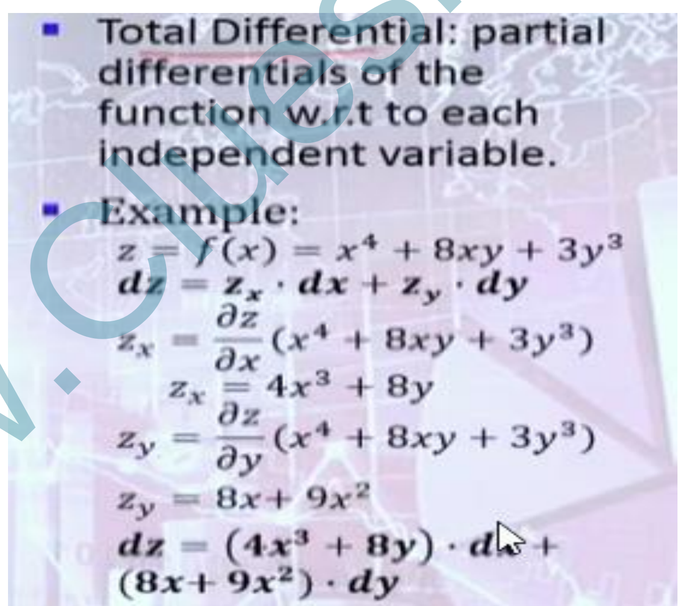

Efficient Production
Buy in Bulk
Reduction in Promotion Costs
Economies of Scale
Reduction in Logistics Costs
Cheaper Capital
Spread Risk

Lesson 28

USE OF TOTAL DIFFERENTIALS IN ECONOMICS

TOPIC 132: CONCEPT OF TOTAL DIFFERENTIALS





TOPIC 133: SAVINGS FUNCTION AND TOTAL DIFFERENTIALS

If $\boldsymbol{S = s(Y, i)}$

S =Savings, Y =National income, i = Interest rate.

Synonymous to:

$\boldsymbol{dz = z_x(x, \bar{y}) \cdot dx + z_y(\bar{x}, y) \cdot dy}$

$$dS = \frac{\partial\{S(Y,i)\}}{\partial Y} \cdot dY + \frac{\partial\{S(Y,i)\}}{\partial i} \cdot di$$

Where, $\frac{\partial\{S(Y,i)\}}{\partial Y} = MPS$

$$dS = S_Y(Y, \bar{\imath}) \cdot dY + S_i(\bar{Y}, i) \cdot di$$

$$dS = S_Y \cdot dY + S_i \cdot di$$

Total Differential: Sum of partial differentials of the function w.r.t to each independent variable.

Example

If $S(Y,i) = \frac{1}{2}Y + 3i$

$S_Y = \frac{\partial}{\partial Y}\left(\frac{1}{2}Y + 3i\right) = \frac{1}{2}$

$S_i = \frac{\partial}{\partial i}\left(\frac{1}{2}Y + 3i\right) = 3$

$dS = S_Y \cdot dY + S_i \cdot di$

$dS = \frac{1}{2} \cdot dY + 3 \cdot di$

$dS = \frac{1}{2} \cdot dY + 3 \cdot di$

$\frac{1}{2} \cdot dY$: Change in savings due to change in income is half of it.

$3 \cdot di$: Change in savings due to change in interest rate is thrice of it.

TOPIC 134: GENERAL UTILITY FUNCTION AND TOTAL DIFFERENTIALS

In real world, utility depends on multiple goods instead of 2.
Case of n-goods utility function.

$$U = U(x_1, x_2, \ \dots, x_n)$$

Total differential dU:

$dU = \left(\frac{\partial U}{\partial x_1}\right) dx_1 + \left(\frac{\partial U}{\partial x_2}\right) dx_2 + \cdots + \left(\frac{\partial U}{\partial x_n}\right) dx_n$

$dy = U_1 dx_1 + U_2 dx_2 + \cdots + U_n dx_n$

$dy = \sum_{i=1}^{n} U_i \, dx_i$

Total differential is composed of partial differentials.

In case of n-goods utility function $U = U(x_1, x_2, \ \dots, x_n)$, there will be n-partial differentials:

$\left(\frac{\partial U}{\partial x_1}\right) dx_1, \left(\frac{\partial U}{\partial x_2}\right) dx_2, \dots, \left(\frac{\partial U}{\partial x_n}\right) dx_n$

Borrowing the partial derivative expressions.

$\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \dots, \frac{\partial U}{\partial x_n}$.

Multiple partial elasticities can be developed from these:

$\varepsilon_{Ux_1} = \left(\frac{\partial U}{\partial x_1}\right)\left(\frac{x_1}{U}\right)$

$\varepsilon_{Ux_2} = \left(\frac{\partial U}{\partial x_2}\right)\left(\frac{x_2}{U}\right)$

…………………………………………

$\varepsilon_{Ux_n} = \left(\frac{\partial U}{\partial x_2}\right)\left(\frac{x_n}{U}\right)$

General form of partial elasticities of utility functions:

$\varepsilon_{Ux_i} = \left(\frac{\partial U}{\partial x_i}\right)\left(\frac{x_i}{U}\right)$

TOPIC 135: SPECIFIC UTILITY FUNCTION AND TOTAL DIFFERENTIALS

(1) $U(x_1, x_2) = ax_1 + bx_2$

Since U is dependent on two variables, we find the total differential.

$$dU(x_1, x_2) = U_1 \cdot dx_1 + U_2 \cdot dx_2$$

$$U_1 = \frac{\partial U}{\partial x_1} = \frac{\partial}{\partial x_1}(ax_1 + bx_2)$$

$$U_1 = a$$

$$U_2 = \frac{\partial U}{\partial x_2} = \frac{\partial}{\partial x_2}(ax_1 + bx_2)$$

$$U_2 = b$$

Substituting values in total differential formula.

$$dU(x_1, x_2) = a \cdot dx_1 + b \cdot dx_2$$

(2) $U(x_1, x_2) = x_1^2 + x_2^3 + x_1 x_2$

Total differential

$$dU(x_1, x_2) = U_1 \cdot dx_1 + U_2 \cdot dx_2$$

$$U_1 = \frac{\partial U(x_1, x_2)}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2 + x_2^3 + x_1 x_2)$$

$$U_1 = 2x_1 + x_2$$

$$U_2 = \frac{\partial}{\partial x_2} U(x_1, x_2) = \frac{\partial}{\partial x_2}(x_1^2 + x_2^3 + x_1 x_2)$$

$$U_2 = 3x_2^2 + x_1$$

$$dU(x_1, x_2) = (2x_1 + x_2)\, dx_1 + (3x_2^2 + x_1)\, dx_2$$

(3) $U(x_1, x_2) = x_1^a x_2^b$

Total differential.

$$dU(x_1,x_2) = U_1 \cdot dx_1 + U_2 \cdot dx_2.$$

$$U_1 = \frac{\partial}{\partial x_1} U(x_1,x_2) = \frac{\partial}{\partial x_1}(x_1^a x_2^b)$$

$$U_1 = a x_1^{a-1} \cdot x_2^b = \frac{a x_1^a x_2^b}{x_1}$$

$$U_2 = b x_1^a \cdot x_2^{b-1} = \frac{b x_1^a x_2^b}{x_2}$$

$$dU(x_1,x_2) = \left(\frac{a x_1^a x_2^b}{x_1}\right) dx_1 + \left(\frac{b x_1^a x_2^b}{x_2}\right) dx_2$$

TOPIC 136: PRICE AND RAIN ELASTICITY OF SUPPLY USING TOTAL DIFFERENTIALS

Supply function of a certain commodity is:

$$Q = a + bP^2 + R^{1/2} \quad (a < 0, \ b > 0)$$

R = Rainfall

(i) Price elasticity of demand:

$$\epsilon_p = \frac{\partial Q / \partial P}{Q / P}$$

$$\frac{\partial(Q)}{\partial P} = \frac{\partial}{\partial P}(a + bP^2 + R^{1/2})$$

$$= 0 + 2bP + 0$$

$$\frac{\partial Q}{\partial P} = 2bP$$

$$= \frac{2bP}{\frac{a + bP^2 + R^{1/2}}{P}}$$

Reciprocalizing the denominator.

$$= \frac{2bP \times P}{a + bP^2 + R^{1/2}}$$

$$\epsilon_p = \frac{2bP^2}{a + bP^2 + R^{1/2}}$$

② Rainfall Elasticity of demand:

$$\epsilon_R = \frac{\partial Q/\partial R}{Q/R}$$

$$\frac{\partial}{\partial R}(Q) = \frac{\partial}{\partial R}(a + bP^2 + R^{1/2})$$

$$= 0 + 0 + \frac{1}{2}R^{-1/2}$$

$$\frac{dQ}{\partial R} = \frac{1}{2\sqrt{R}}$$

$$= \frac{\frac{1}{2\sqrt{R}}}{\frac{a + bP^2 + R^{1/2}}{R}}$$

$$= \frac{1 \times R}{2\sqrt{R}(a + bP^2 + R^{1/2})}$$

$$= \frac{R}{\sqrt{R}} \times \frac{1}{2(a + bP^2 + R^{1/2})}$$

$$\boxed{\epsilon_R = \frac{\sqrt{R}}{2(a + bP^2 + R^{1/2})}}$$

TOPIC 137: LOCAL PRICE ELASTICITY OF FOREIGN DEMAND OF EXPORTS USING TOTAL DIFFERENTIALS

Considering foreign demand for our exports (X):

$$X = Y_f^{1/2} + P^{-2}$$

Y_f = foreign income

P = local price level.

① Price elasticity of foreign demand (ϵ_{XP}).

$$\epsilon_{XP} = \frac{\partial X/\partial P}{X/P}$$

$$\frac{\partial}{\partial P}(X) = \frac{\partial}{\partial P}(Y_f^{1/2} + P^{-2})$$

$$= 0 + (-2)P^{-3}$$

$$\frac{\partial X}{\partial P} = \frac{-2}{P^3}$$

$$= \frac{-2/P^3}{\frac{Y_f^{1/2} + P^{-2}}{P}}$$

$$= \frac{-2 \cdot P}{P^3(Y_f^{1/2} + P^{-2})}$$

$$\epsilon_{XP} = \frac{-2}{P^2(Y_f^{1/2} + P^{-2})} = \frac{-2}{Y_f^{1/2} \cdot P^2 + 1}$$

② Foreign Income elasticity of foreign demand

$$\epsilon_{xF} = \frac{\partial x / \partial Y_f}{x / Y_f}$$

$$\frac{\partial (x)}{\partial Y_f} = \frac{\partial}{\partial Y_f}\left(Y_f^{1/2} + P^{-2}\right)$$

$$= \frac{1}{2\sqrt{Y_f}} + 0$$

$$\boxed{\frac{\partial x}{\partial Y_f} = \frac{1}{2\sqrt{Y_f}}}$$

$$= \frac{\left(\frac{1}{2\sqrt{Y_f}}\right)}{\frac{Y_f^{1/2} + P^{-2}}{Y_f}}$$

$$= \frac{1}{2\sqrt{Y_f}} \times \frac{Y_f}{\left(Y_f^{1/2} + P^{-2}\right)}$$

$$\boxed{\epsilon_{xF} = \frac{\sqrt{Y_f}}{2\left(Y_f^{1/2} + P^{-2}\right)}}$$

Lesson 29

CONCEPT OF TOTAL DERIVATIVES

TOPIC 138: CONCEPT OF TOTAL DERIVATIVE

Assume $y = f(x, w)$ Where, $x = g(w)$.
Channel map.

$$\begin{array}{ccccc} y & \overset{f}{\Leftarrow} & x & \overset{g}{\Leftarrow} & w \\ \nwarrow & \leftarrow & \overset{f}{\Leftarrow} & \leftarrow & \swarrow \end{array}$$

Combining the two functions

$$y = f\{g(w), w\}$$

w: Ultimate source of change.

$$y = f\{g(w), w\}$$

Two channels: Direct & Indirect.

- Indirect: via function g.
- Direct: via function f.

- Indirect: $y = f\{g(w)\} \Rightarrow [y\{x(w)\}] \Rightarrow \frac{\partial y}{\partial x} \cdot \frac{dx}{dw} = f_x \cdot \frac{dx}{dw}$
- Direct: $y = f(w) \Rightarrow \frac{\partial y}{\partial w} = f_w$

Total (Direct +Indirect): $y = f\{g(w), w\}$

$$\underbrace{\frac{dy}{dw}}_{\substack{Total\\Effect}} = \underbrace{f_x \cdot \frac{dx}{dw}}_{\substack{Indirect\\Effect}} + \underbrace{f_w}_{\substack{Direct\\Effect}}$$

Total Differentiation of y w.r.t. w.

Caveat:

$$\frac{dy}{dw} = f_x \cdot \frac{dx}{dw} + f_w$$

$$\underbrace{\frac{dy}{dw}}_{\substack{Total\\Derivative\\of\ y\ w.r.t.\ w.}} = \frac{\partial y}{\partial x} \cdot \frac{dx}{dw} + \underbrace{\frac{\partial y}{\partial w}}_{\substack{Partial\\Derivative\\of\ y\ w.r.t.\ w.}}$$

Example

$$y = f(x, w) = 3x - w^2$$
$$x = g(w) = 2w^2 + w + 4$$
$$\frac{dy}{dw} = f_x \cdot \frac{dx}{dw} + f_w$$

$f_x = \frac{\partial f}{\partial x}(3x - w^2) = 3$

$\frac{dx}{dw} = \frac{d}{dw}(2w^2 + w + 4) = 4w + 1$

$f_w = \frac{\partial y}{\partial w}(3x - w^2) = -2w$

$\frac{dy}{dw} = 3(4w + 1) + (-2w)$

$\frac{dy}{dw} = 10w + 3$

Verification

$$y = f(x, w) = 3x - w^2$$

$$x = g(w) = 2w^2 + w + 4$$

Substituting:

$$y = h(w) = 3(2w^2 + w + 4) - w^2$$

$y = h(w) = 5w^2 + 3w + 12$

Differentiating w.r.t. w.

$\frac{dy}{dw} = \frac{d}{dw}\{h(w)\} = \frac{d}{dw}\{5w^2 + 3w + 12\}$

$\frac{dy}{dw} = \mathbf{10w + 3}$ (Same as that in Total Derivative formula).

TOPIC 139: COMPLEMENTARITY BETWEEN COFFEE AND SUGAR USING TOTAL DERIVATIVE

Assume $\boldsymbol{U = u(c, s)}$

Where $\boldsymbol{c}$ and $\boldsymbol{s}$ are coffee and sugar, respectively.

Furthermore; $\boldsymbol{s = g(c)}$.

This implies complementarity between sugar and coffee (Financial Times newspaper).

$\boldsymbol{U = u\{c, g(c)\}}$

Channel map:

$$\begin{matrix} \boldsymbol{U} & \overset{u}{\Leftarrow} & \boldsymbol{s} & \overset{g}{\Leftarrow} & \boldsymbol{c} \\ \nwarrow & \leftarrow & \boldsymbol{u} & \leftarrow & \swarrow \end{matrix}$$

Main variable causing change is $\boldsymbol{c}$:

$$\boldsymbol{U = u\{c, g(c)\}}$$

$$\boldsymbol{\frac{dU}{dc} = \frac{\partial U}{\partial c} + \frac{\partial U}{\partial s} \cdot \frac{ds}{dc}}$$

$$\boldsymbol{\frac{dU}{dc} = U_c + U_s \cdot \frac{ds}{dc}}$$

Example

$$\boldsymbol{U = u\{c,\ s\} = 6c^3 + 7s}$$

$$\boldsymbol{s = g(c) = 4c^2 + 3c + 8}$$

$$\boldsymbol{U_c = \frac{\partial}{\partial c}\{u(c,\ s)\} = 18c^2}$$

$$\boldsymbol{U_s = \frac{\partial}{\partial s}\{u(c,\ s)\} = 7}$$

$$\boldsymbol{\frac{ds}{dc} = \frac{d}{dc}\{g(c)\} = 8c + 3}$$

$$\boldsymbol{\frac{dU}{dc} = U_c + U_s \cdot \frac{ds}{dc}}$$

$$\boldsymbol{\frac{dU}{dc} = 18c^2 + 7 \cdot (8c + 3)}$$

$$\boldsymbol{\frac{dU}{dc} = 18c^2 + 56c + 21}$$

Verify using substitution.

TOPIC 140: GENERAL PRODUCTION FUNCTION WITH TIME-DEPENDENT LABOR AND CAPITAL

Consider a production function.

$Q = Q(K, L, t)$

Q = output, K = Capital, L = labour

t = time.

- time (t) shows that technological changes can occur, shifting the production outward.
- Therefore, the production function is a dynamic production function rather than static. i.e. $Q(t)$.
- Moreover, Capital and labour can also improve over time i.e. Capital can become improved physical capital over time. Labour can become improved human capital over time

i.e. $K(t)$ & $L(t)$

$\Rightarrow Q = Q\{K(t), L(t), t\}$

Dependence of Q on t can be written using partial derivative.

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Q}{\partial L} \cdot \frac{dL}{dt} + \frac{\partial Q}{\partial t}$$

$$= Q_K \cdot \frac{dK}{dt} + Q_L \cdot \frac{dL}{dt} + Q_t$$

$$\boxed{\frac{\partial Q}{\partial t} = Q_K \cdot K'(t) + Q_L \cdot L'(t) + Q_t}$$

TOPIC 141: SPECIFIC PRODUCTION FUNCTION WITH TIME-DEPENDENT LABOR AND CAPITAL

Consider a dynamic production function

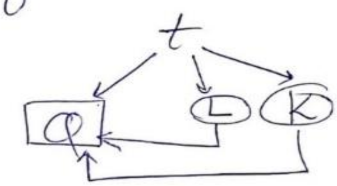
$$Q = A(t) \cdot K^{\alpha} L^{\beta}$$

$A(t)$ = increasing function of time (t)

$K = K_0 + at$ $\quad K_0$ = initial condition of K, at = time-induced improvement in capital

$L = L_0 + bt$ $\quad L_0$ = initial condition of L, bt = time-induced improvement in labour.

Impact of time (t) on output is two-Tiered



$$\frac{\partial Q}{\partial t} = Q_t + Q_K \cdot K'(t) + Q_L \cdot L'(t)$$

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Q}{\partial L} \cdot \frac{dL}{dt}$$

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}\left(A(t) \cdot K^{\alpha} L^{\beta} \right), \quad \frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}\left(A(t) \cdot K^{\alpha} L^{\beta} \right)$$

$$\frac{\partial Q}{\partial t} = A'(t) \cdot K^{\alpha} L^{\beta} \qquad = A(t) \cdot \alpha K^{\alpha-1} L^{\beta}$$

$$\frac{\partial Q}{\partial K} = \alpha A(t) \cdot K^{\alpha-1} L^{\beta}$$

$$\frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}\left\{ A(t) K^{\alpha} L^{\beta} \right\}.$$

$$= A(t) \cdot K^{\alpha} \cdot \beta L^{\beta-1}$$

$$\frac{\partial Q}{\partial L} = \beta A(t) \cdot K^{\alpha} L^{\beta-1}$$

$$\frac{dK}{dt} = \frac{d}{dt}(K_0 + at) \qquad \frac{dL}{dt} = \frac{d}{dt}(L_0 + bt)$$

$$\frac{dK}{dt} = a \qquad \frac{dL}{dt} = b$$

Substituting all values:

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Q}{\partial L} \cdot \frac{dL}{dt}$$

$$= A'(t) \cdot K^{\alpha} L^{\beta} + \alpha \cdot A(t) K^{\alpha-1} L^{\beta} \cdot (a) + \beta A(t) \cdot K^{\alpha} L^{\beta-1} \cdot (b)$$

$$= A'(t) K^{\alpha} L^{\beta} + a\alpha A(t) \cdot K^{\alpha-1} L^{\beta} + b\beta \cdot A(t) K^{\alpha} L^{\beta-1}$$

$$= K^{\alpha} L^{\beta} \left\{ A'(t) + a\alpha A(t) \cdot K^{-1} + b\beta A(t) L^{-1} \right\}$$

$$\frac{\partial Q}{\partial t} = K^{\alpha} L^{\beta} \left\{ A'(t) + \frac{a\alpha A(t)}{K} + \frac{b\beta A(t)}{L} \right\}$$

Lesson 30

CONCEPT OF IMPLICIT DIFFERENTIATION AND THEIR ECONOMIC APPLICATIONS

TOPIC 142: CONCEPT OF IMPLICIT DIFFERENTIATION

Deals with implicit functions:

- $8x + 5y = 21$
- $3x^2 - 8xy - 5y = 49$
- $35x^3y^7 = 106$

To find $\frac{dy}{dx}$ of implicit function:

- Differentiate each side of the equation w.r.t. x, considering y as a function of x.
- Solve the resulting equation for $\frac{dy}{dx}$.

Example

$$3x^2 - 8xy - 5y = 49$$
$$d/_{dx}(3x^2 - 8xy - 5y) = d/_{dx}(49)$$
$$\frac{d}{dx}(3x^2) - 8\frac{d}{dx}(xy) - 5\frac{d}{dx}(y) = 0$$
$$3(2x)\frac{dx}{dx} - 8\left\{y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\right\} - 5\frac{dy}{dx} = 0$$
$$6x\frac{dx}{dx} - 8\left\{y\frac{dx}{dx} + x\frac{dy}{dx}\right\} - 5\frac{dy}{dx} = 0$$
$$6x - 8\left\{y + x\frac{dy}{dx}\right\} - 5\frac{dy}{dx} = 0$$
$$6x - 8y - 8x\frac{dy}{dx} - 5\frac{dy}{dx} = 0$$
$$6x - 8y = 8x\frac{dy}{dx} + 5\frac{dy}{dx}$$
$$6x - 8y = 8x\frac{dy}{dx} + 5\frac{dy}{dx}$$
$$6x - 8y = (8x + 5)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{6x - 8y}{8x + 5}$$

Implicit derivative of the given implicit function.

TOPIC 143: PRODUCTION FUNCTION ANALYSIS USING IMPLICIT DIFFERENTIATION

Assume

$$Q = f(K, L)$$

Implicit function version.

$$F(Q, K, L) = 0$$

Marginal physical products of labor & capital (MPP_K & MPP_L).

These are partial derivatives ($\frac{\partial}{\partial K}$ & $\frac{\partial}{\partial L}$).

$MPP_K = \frac{\partial}{\partial K}\{F(Q, K, \bar{L})\} = \frac{\partial}{\partial K}\{0\}\ \frac{\partial}{\partial K}\{F(Q(K), K, \bar{L})\} = 0$

Resorting to implicit differentiation: $\frac{\partial}{\partial K}\{F(Q(K), K, \bar{L})\} = 0$

$$\frac{\partial F}{\partial Q} \cdot \frac{dQ}{dK} + \frac{\partial F}{\partial K} = 0$$

$$\frac{dQ}{dK} = -\frac{\left(\frac{\partial F}{\partial K}\right)}{\left(\frac{\partial F}{\partial Q}\right)} = -\left(\frac{F_K}{F_Q}\right)$$

$$MPP_K = \frac{dQ}{dK} = -\left(\frac{F_K}{F_Q}\right)$$

Marginal physical product of capital (MPP_K) expressed in relation to the function $F(Q, K, L)$.

Similarly, $MPP_L = \frac{\partial}{\partial L}\{F(Q, \bar{K}, L)\} = \frac{\partial}{\partial L}\{0\}$

$\frac{\partial}{\partial L}\{F(Q(L), \bar{K}, L)\} = 0$

$\frac{\partial F}{\partial Q} \cdot \frac{dQ}{dL} + \frac{\partial F}{\partial L} = 0$

$\frac{dQ}{dL} = -\frac{\left(\frac{\partial F}{\partial L}\right)}{\left(\frac{\partial F}{\partial Q}\right)} = -\left(\frac{F_L}{F_Q}\right)$

$MPP_L = \frac{dQ}{dL} = -\left(\frac{F_L}{F_Q}\right)$

Marginal physical product of labor (MPP_L) expressed in relation to the function $F(Q, K, L)$.

Also $MRTS_{(K,L)}$ can be found using implicit differentiation.

$\frac{\partial}{\partial L}\{F(\bar{Q}, K(L), L)\} = \frac{\partial}{\partial L}\{0\}$

$\frac{\partial F}{\partial K} \cdot \frac{dK}{dL} + \frac{\partial F}{\partial L} = 0$

$\frac{dK}{dL} = -\frac{\left(\frac{\partial F}{\partial L}\right)}{\left(\frac{\partial F}{\partial K}\right)} = -\left(\frac{F_L}{F_K}\right)$

$MRTS_{(K,L)} = \frac{dK}{dL} = -\left(\frac{F_L}{F_K}\right)$

Marginal rate of technical substitution ($MRTS_{K,L}$) expressed in relation to the function $F(Q, K, L)$.

TOPIC 144: MARGINAL RATE OF TECHNICAL SUBSTITUTION USING IMPLICIT DIFFERENTIATION

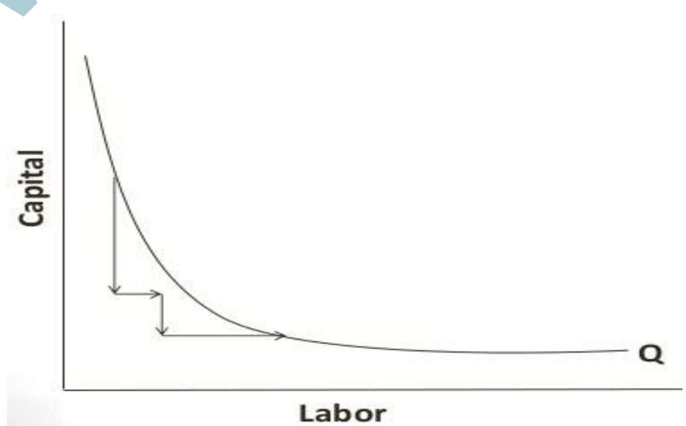
Assume

$F(Q, K, L) = 0$

An implicit function that can yield a production function:

$$Q = f(K, L)$$

Marginal rate of technical substitution, $\Delta Q (= dQ) = 0$



$MRTS_{(K,L)}$: Implicit Differentiation

$\frac{\partial}{\partial L}\{F(\bar{Q}, K(L), L)\} = \frac{\partial}{\partial L}\{0\}$

$\frac{\partial F}{\partial K} \cdot \frac{dK}{dL} + \frac{\partial F}{\partial L} = 0$

$\frac{dK}{dL} = -\frac{\left(\frac{\partial F}{\partial L}\right)}{\left(\frac{\partial F}{\partial K}\right)} = -\left(\frac{F_L}{F_K}\right)$

$MRTS_{(K,L)} = \frac{dK}{dL}\Big|_{Q=0} = -\left(\frac{F_L}{F_K}\right)$

Marginal rate of technical substitution ($MRTS_{K,L}$) expressed in relation to the function $F(Q, K, L)$.

TOPIC 145: MARGINAL UTILITIES AND MARGINAL RATE OF SUBSTITUTION USING IMPLICIT DIFFERENTIATION

Assume an implicit equation:

$$F(U, x_1, x_2, \dots, x_n) = 0$$

Utility function can be extracted:

$$U = f(x_1, x_2, \dots, x_n)$$

For $\frac{\partial U}{\partial x_2}$, assume $F(U)$ & $U(x_2) \Rightarrow F\{U(x_2)\}$ *ceteris paribus* $(U, \bar{x}_1, x_2, \dots, \bar{x}_n)$.

$\frac{\partial\{F(U, x_1, x_2, \dots, x_n)\}}{\partial x_2} = 0$

$\frac{\partial F}{\partial x_2} + \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial x_2} = 0$

Re-arranging

$\frac{\partial U}{\partial x_2} = -\frac{\left(\frac{\partial F}{\partial x_2}\right)}{\left(\frac{\partial F}{\partial U}\right)} = MU_2$

For $\frac{\partial U}{\partial x_n}$, assume

$F(U)$ & $U(x_n) \Rightarrow F\{U(x_n)\}$, *ceteris paribus* $\left(U,\ \bar{x}_1, \bar{x}_2, \dots, x_n\right)$

$\frac{\partial\{F(U, \bar{x}_1, \bar{x}_2, \dots, x_n)\}}{\partial x_n} = 0$

$\frac{\partial F}{\partial x_n} + \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial x_n} = 0$

$\frac{\partial U}{\partial x_n} = -\frac{\left(\frac{\partial F}{\partial x_n}\right)}{\left(\frac{\partial F}{\partial U}\right)} = MU_n$

Marginal utility due to an additional unit of x_n (n^{th} good).

For $\frac{\partial x_3}{\partial x_2}$, assume $F(x_2)$ & $x_2(x_3) \Rightarrow F\{x_2(x_3)\}$, *ceteris paribus* $(\bar{U}, \bar{x}_1, x_2, x_3, \dots, \bar{x}_n)$

$\frac{\partial\{F(\bar{U}, \bar{x}_1, x_2, x_3, \dots, \bar{x}_n)\}}{\partial x_n} = 0$

$\frac{\partial F}{\partial x_3} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_3} = 0$

$-\frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_3}$

$-\frac{\left(\frac{\partial F}{\partial x_3}\right)}{\left(\frac{\partial F}{\partial x_2}\right)} = \frac{\partial x_2}{\partial x_3}$

$$\frac{\partial x_3}{\partial x_2} = -\frac{\left(\frac{\partial F}{\partial x_2}\right)}{\left(\frac{\partial F}{\partial x_3}\right)} = MRS^{IC}_{(x_3,x_2)}$$

For $\frac{\partial x_4}{\partial x_n}$, Assume

$F(x_4)$ & $x_4(x_n) \Rightarrow F\{x_4(x_n)\}$, *ceteris paribus* $\left(\bar{U}, \bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \ \ldots, x_n\right)$

$$\frac{\partial\{F\left(\bar{U},\bar{x}_1,\bar{x}_2,\bar{x}_3,x_4, \ldots,x_n\right)\}}{\partial x_n} = 0$$

$$\frac{\partial F}{\partial x_n} + \frac{\partial F}{\partial x_4}\cdot\frac{\partial x_4}{\partial x_n} = 0$$

$$-\frac{\partial F}{\partial x_n} = \frac{\partial F}{\partial x_4}\cdot\frac{\partial x_4}{\partial x_n}$$

$$-\frac{\left(\frac{\partial F}{\partial x_n}\right)}{\left(\frac{\partial F}{\partial x_4}\right)} = \frac{\partial x_4}{\partial x_n}$$

$$\frac{\partial x_4}{\partial x_n} = -\frac{\left(\frac{\partial F}{\partial x_n}\right)}{\left(\frac{\partial F}{\partial x_4}\right)} = MRS^{IC}_{(x_4,x_n)}$$

TOPIC 146: NERLOVE-RINGSTAD PRODUCTION FUNCTION USING IMPLICIT DIFFERENTIATION

Attributed to Nerlove (1963) and Ringstad (1967).

$$y^{1+c\ln|y|} = AK^{\alpha}L^{\beta}$$

Where, $A > 0$, $\alpha > 0$ and $\beta > 0$.

$$\ln\left|y^{1+c\ln|y|}\right| = \ln\left|AK^{\alpha}L^{\beta}\right|$$

$$\ln\left|y^{1+c\ln|y|}\right| = \ln|A| + \ln|K^{\alpha}| + \ln\left|L^{\beta}\right|$$

$$(1 + c\ln|y|)(\ln|y|) = \ln|A| + \ln|K^{\alpha}| + \ln\left|L^{\beta}\right|$$

y is not in explicit function form.
Implicit differentiation w.r.t. K

$$\frac{d}{dK}\{(1 + c\ln|y(K,\bar{L})|)(\ln|y(K,\bar{L})|)\} = \frac{d}{dK}\left(\ln|A| + \ln|K^{\alpha}| + \ln\left|L^{\beta}\right|\right)$$

$$\frac{d}{dK}\left\{\left(1 + c\ln|y(K,\bar{L})|\right)\right\}(\ln|y(K,\bar{L})|) + \frac{d}{dK}\{(\ln|y(K,\bar{L})|)\}\left(1 + c\ln|y(K,\bar{L})|\right) = 0 + \frac{d}{dK}(\ln|K^{\alpha}|) + 0$$

$$\left[c\cdot\frac{\frac{d}{dK}\{y(K,\bar{L})\}}{y}\right](\ln|y(K,\bar{L})|) + \left[\frac{\frac{d}{dK}\{y(K,\bar{L})\}}{y}\right]\left(1 + c\ln|y(K,\bar{L})|\right) = \frac{\frac{d}{dK}(K^{\alpha})}{K^{\alpha}}$$

$$\frac{d}{dK}\{y(K,\bar{L})\}\cdot\left[\frac{c\ln|y(K,\bar{L})|}{y}\right] + \left[\frac{(1+c\ln|y(K,\bar{L})|)}{y}\right]\frac{d}{dK}\{y(K,\bar{L})\} = \frac{\alpha K^{\alpha-1}}{K^{\alpha}}$$

$$\frac{d}{dK}\{y(K,\bar{L})\}\cdot\left[\frac{c\ln|y(K,\bar{L})|}{y} + \frac{(1+c\ln|y(K,\bar{L})|)}{y}\right] = \alpha K^{-1}$$

Writing in simpler notation.

$$\frac{d}{dK}(y)\cdot\left[\frac{c\ln|y|}{y} + \frac{(1+c\ln|y|)}{y}\right] = \frac{\alpha}{K}$$

$$\frac{dy}{dK}\cdot\left[\frac{c\,ln|y|+1+c\,ln|y|}{y}\right]=\frac{\alpha}{K}$$
$$\frac{dy}{dK}\cdot\left[\frac{1+2c\ln|y|}{y}\right]=\frac{\alpha}{K}$$
$$\frac{dy}{dK}=MP_K=\frac{\alpha y}{K(1+2c\;\ln|y|)}$$

Implicit differentiation w.r.t. L.

$$\frac{d}{dL}\{(1+c\,ln|y(\bar{K},L)|)(\ln|y(\bar{K},L)|)\}=\frac{d}{dL}\left(\ln|A|+\ln|K^\alpha|+\ln\left|L^\beta\right|\right)$$
$$\frac{d}{dL}\{(1+c\,ln|y(\bar{K},L)|)\}(\ln|y(\bar{K},L)|)+\frac{d}{dL}\{(\ln|y(\bar{K},L)|)\}(1+c\,ln|y(\bar{K},L)|)=\frac{d}{dL}\left(\ln\left|L^\beta\right|\right)$$
$$\left[c\cdot\frac{\frac{d}{dL}\{y(\bar{K},L)\}}{y}\right](\ln|y(\bar{K},L)|)+\left[\frac{\frac{d}{dL}\{y(\bar{K},L)\}}{y}\right](1+c\,ln|y(\bar{K},L)|)=\frac{\frac{d}{dL}(L^\beta)}{L^\beta}$$

$$\frac{d}{dL}\{y(\bar{K},L)\}\cdot\left[\frac{c\,ln|y(\bar{K},L)|}{y}\right]+\left[\frac{(1+c\,ln|y(\bar{K},L)|)}{y}\right]\frac{d}{dL}\{y(\bar{K},L)\}=\frac{\beta L^{\beta-1}}{L^\beta}$$
$$\frac{d}{dL}\{y(\bar{K},L)\}\cdot\left[\frac{c\,ln|y(\bar{K},L)|}{y}+\frac{(1+c\,ln|y(\bar{K},L)|)}{y}\right]=\beta L^{-1}$$

Writing in simpler notation.

$$\frac{d}{dL}(y)\cdot\left[\frac{c\,ln|y|}{y}+\frac{(1+c\ln|y|)}{y}\right]=\frac{\beta}{L}$$
$$\frac{dy}{dL}\cdot\left[\frac{c\,ln|y|+1+c\,ln|y|}{y}\right]=\frac{\beta}{L}$$
$$\frac{dy}{dL}\cdot\left[\frac{1+2c\ln|y|}{y}\right]=\frac{\beta}{L}$$
$$\frac{dy}{dL}=MP_L=\frac{\beta y}{L(1+2c\;\ln|y|)}$$

TOPIC 147: MARGINAL PRODUCTS OF THREE INPUT LOGARITHMIC PRODUCTION FUNCTION

Assume an endogenous growth model:

$$Q=A\,L^\alpha K^\beta H^\gamma$$

H: Human capital

Becker (1964): Skills and adequate motivation to apply them.

Logarithmically, linearizing the production function.

$$\ln|Q|=\ln|A|+\alpha\ln|L|+\beta\ln|K|+\gamma\ln|H|$$

For MP_L, MP_K & MP_H, partially differentiation w.r.t logarithms of L, K, & H.

$$\frac{\partial}{\partial\ln|L|}\{\ln|Q|\}=\frac{\partial}{\partial\ln|L|}\{\ln|A|+\alpha\ln|L|+\beta\ln|K|+\gamma\ln|H|\}$$
$$=\frac{\partial\ln|A|}{\partial\ln|L|}+\frac{\partial(\alpha\ln|L|)}{\partial\ln|L|}+\frac{\partial(\beta\ln|K|)}{\partial\ln|L|}+\frac{\partial(\gamma\ln|H|)}{\partial\ln|L|}$$
$$=\frac{\partial\ln|A|}{\partial\ln|L|}+\alpha\frac{\partial(\ln|L|)}{\partial\ln|L|}+\beta\frac{\partial(\ln|K|)}{\partial\ln|L|}+\gamma\frac{\partial(\ln|H|)}{\partial\ln|L|}$$

$$=\frac{\partial\ln|A|}{\partial\ln|L|}+\alpha\frac{\partial(\ln|L|)}{\partial\ln|L|}+\beta\frac{\partial(\ln|K|)}{\partial\ln|L|}+\gamma\frac{\partial(\ln|H|)}{\partial\ln|L|}$$
$$=0+\alpha(1)+\beta(0)+\gamma(0)$$

$\boldsymbol{MP_L = \frac{\partial}{\partial \ln|L|}\{\ln|Q|\} = \alpha}$ (Labor elasticity of output)

$$\frac{\partial \ln|Q|}{\partial \ln|L|} = \frac{\%\Delta|Q|}{\%\Delta|L|}$$

For MP_K:

$$\frac{\partial}{\partial \ln|K|}\{\ln|Q|\} = \frac{\partial}{\partial \ln|K|}\{\ln|A| + \alpha \ln|L| + \beta \ln|K| + \gamma \ln|H|\}$$

$$= \frac{\partial \ln|A|}{\partial \ln|K|} + \frac{\partial(\alpha \ln|L|)}{\partial \ln|K|} + \frac{\partial(\beta \ln|K|)}{\partial \ln|K|} + \frac{\partial(\gamma \ln|H|)}{\partial \ln|K|}$$

$$= \frac{\partial \ln|A|}{\partial \ln|K|} + \alpha\frac{\partial(\ln|L|)}{\partial \ln|K|} + \beta\frac{\partial(\ln|K|)}{\partial \ln|K|} + \gamma\frac{\partial(\ln|H|)}{\partial \ln|K|} = \beta$$

$MP_K = \beta$ (Capital elasticity of output)

$$MP_H = \frac{\partial\{\ln|Q|\}}{\partial \ln|H|}$$

$$= \frac{\partial \ln|A|}{\partial \ln|H|} + \frac{\partial(\alpha \ln|L|)}{\partial \ln|H|} + \frac{\partial(\beta \ln|K|)}{\partial \ln|H|} + \frac{\partial(\gamma \ln|H|)}{\partial \ln|H|}$$

$$= \frac{\partial \ln|A|}{\partial \ln|H|} + \alpha\frac{\partial(\ln|L|)}{\partial \ln|H|} + \beta\frac{\partial(\ln|K|)}{\partial \ln|H|} + \gamma\frac{\partial(\ln|H|)}{\partial \ln|H|}$$

$\boldsymbol{= 0 + \alpha(0) + \beta(0) + \gamma(0) = \gamma = MP_H = \frac{\partial\{\ln|Q|\}}{\partial \ln|H|}}$ (Human capital elasticity of output)

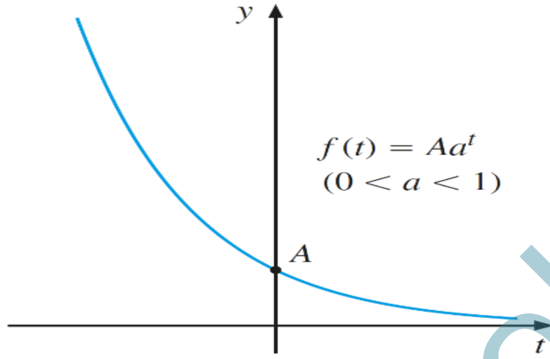
Lesson 31

EXPONENTIAL FUNCTIONS AND GROWTH

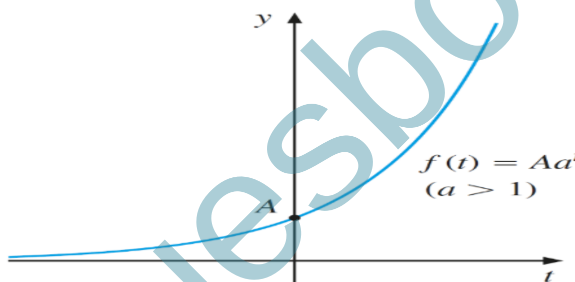
TOPIC 148: EXPONENTIAL FUNCTIONS AND GROWTH

A quantity that increases (or decreases) by a fixed factor per unit of time is said to increase (or decrease) exponentially.

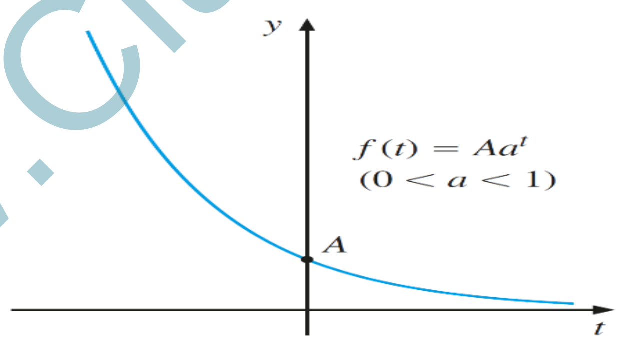
$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{a}^{\boldsymbol{t}}$ Where, $\boldsymbol{a}$ is a factor by which $\boldsymbol{y}$ increases when $\boldsymbol{t}$ increases by 1.



$f(t) = Aa^t,\ 0 < a < 1$



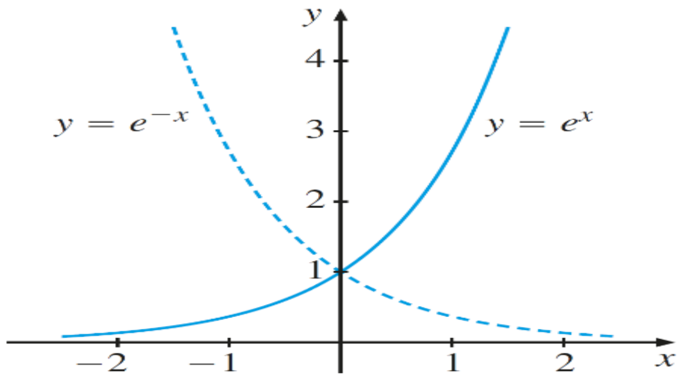
$f(t) = Aa^t,\ a > 1$



$f(t) = Aa^t,\ 0 < a < 1$

Each base $\boldsymbol{a}$ gives a different value of $= \boldsymbol{f}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{a}^{\boldsymbol{t}}$ **.** E.g. $\boldsymbol{a} = \mathbf{2}, \boldsymbol{a} = \mathbf{10}$ etc.
In calculus, most of exponential functions appear with base $\mathbf{2.718281828459045}$ ...

An irrational number.
Represented by $\boldsymbol{e}$**.**
Its function is $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{t}) = \boldsymbol{e}^{\boldsymbol{t}}$ a.k.a. Natural exponential function.



graphs of $y = e^x$ and $y = e^{-x}$

Exemplifies growth of a sum of money capital over time.
Can also represent growth of population, wealth, or real capital etc.

TOPIC 149: INSTANTANEOUS RATE OF GROWTH

Given a value function of continuous interest compounding

$$V = Ae^{rt}$$

$\boldsymbol{t}$:Points in time, $\boldsymbol{A}$: Principal amount, $\boldsymbol{e}$:Natural base, $\boldsymbol{r}$: Interest rate, $\boldsymbol{V}$**:** Value at point in time.

$$\frac{d}{dt}\{V(t)\} = \frac{d}{dt}(Ae^{rt})$$

$= A\{\frac{d}{dt}(e^{rt})\}$=$A\{r(e^{rt})\}$= $r(Ae^{rt}) = r(V)$

Instantaneous Rate of Change

For Instantaneous Growth Rate.

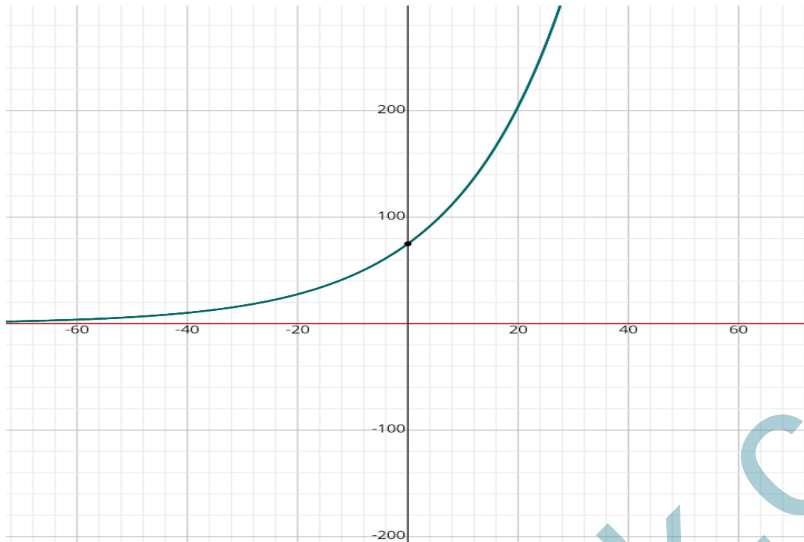
- We find the rate of change of value in relative (%age) terms:
- $= \frac{\frac{d}{dt}\{V(t)\}}{V} = \frac{r(V)}{V} = r$
- **Instantaneous Rate of Growth**

Time dependent.
Constant $\boldsymbol{r}$ (simplicity sake).

Numerical Example

- $y(t) = 75e^{0.02t}$

Instantaneous Rate of Growth $\boldsymbol{r}$ of y is $\mathbf{0.02}$ **–** a constant.



TOPIC 150: NUMERICAL EXAMPLES OF INSTANTANEOUS RATE OF GROWTH

Given:

$$y(t) = e^{0.07t}$$

Interest compounding:

$$V = Ae^{rt}$$

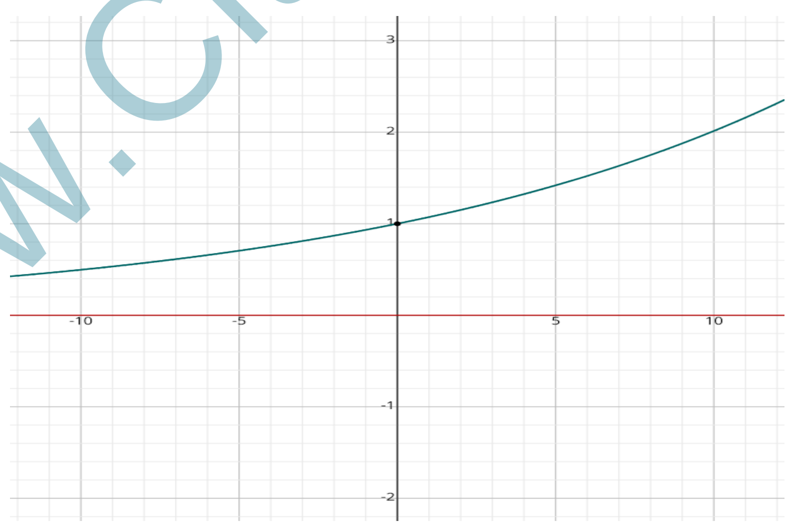
A: Principal amount = 1

G_y: Instantaneous growth rate

$G_y = \frac{\frac{d}{dt}\{y(t)\}}{y}$

$\frac{d}{dt}\{y(t)\} = \frac{d}{dt}\left(e^{0.07t}\right) = 0.07\left(e^{0.07t}\right)$

$$G_y = \frac{0.07\left(e^{0.07t}\right)}{e^{0.07t}} = 0.07 \text{ or } 7\%$$



Given:

$$y(t) = 0.03e^{t}$$

Interest compounding:

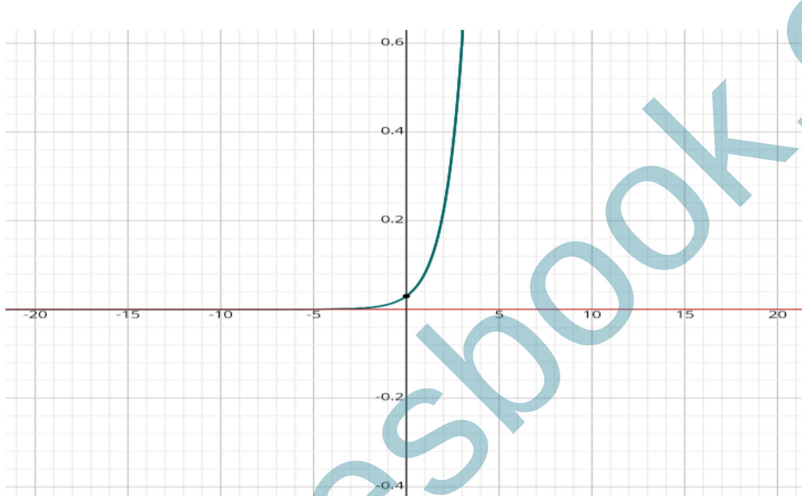
$$V = Ae^{rt}$$

$\boldsymbol{A}$: Principal amount = 0.03,
$\boldsymbol{G_y}$: Instantaneous growth rate

$$G_y = \frac{\frac{d}{dt}\{y(t)\}}{y}$$

$$\frac{d}{dt}\{y(t)\} = \frac{d}{dt}(0.03e^{t}) = 0.03\frac{d}{dt}(e^{t}) = 0.03(e^{t})$$

$$G_y = \frac{0.03(e^{t})}{0.03e^{t}} = 1 \text{ or } 100\%$$



TOPIC 151: CONTINUOUS VS DISCRETE GROWTH

Usually in economic situations, growth does not always take place on a continuous basis.
Not even in interest compounding.
However, for discrete growth, where changes occur only once per period rather than from instant to instant, the assumption of continuous exponential growth function can be justified.

$$A(1+i)^{0},\ A(1+i)^{1}, A(1+i)^{2}, A(1+i)^{3}, \ldots$$

Exponents are time periods covered in compounding, $\boldsymbol{A}$: Principal amount, $\boldsymbol{i}$: Interest rate.
Series can be considered an exponential expression as Ab^{t}.
That is $b = 1 + i$. $b > 0$ even for $i < 0$ as $i < 1$(%age terms).
To convert base to natural exponent, compare Ab^{t} with standard form of natural exponential function Ae^{rt}:

$$Ab^{t} = Ae^{rt} \Rightarrow (b = e^{r})$$

$$1 + i = b = e^{r}$$

$$1 + i = b = e^{r} \Rightarrow$$

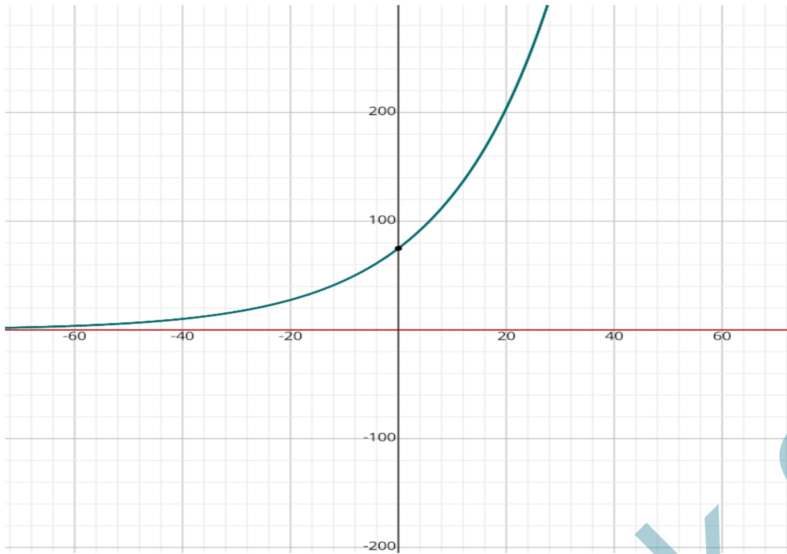
$$A(1+i)^{t} = Ab^{t} = Ae^{rt}$$

$$\Rightarrow A(1+i)^{0} = Ab^{0} = A$$

$$\Rightarrow A(1+i)^{1} = Ab^{1} = Ae^{r}$$

$$\Rightarrow A(1+i)^{2} = Ab^{2} = Ae^{2r}$$

That's why natural exponential functions are extensively applied in economic analysis despite not all growth patterns may actually be (purely) continuous.



TOPIC 152: DISCOUNTING AND NEGATIVE GROWTH

From interest compounding to discounting.
Compound-interest problem: future value **V** (Principal + Interest) from a given present value **A** (Initial principal).
Discounting problem: Present value **A** of a given sum **V** which will be available after **t** years.

- **Discrete case:**

$$V = A(1+i)^t$$

Cross-multiplying:

$\frac{V}{(1+i)^t} = A$

$A = \frac{V}{(1+i)^t}$

$A = V(1+i)^{-t}$

Reversed roles (as in intro.) of ***A*** and ***V.***

- **Continuous Case:**

Principal **A** grows into $V = Ae^{rt}$ after t years of continuous compounding at the rate r .

$$V = Ae^{rt}$$

Cross multiplying:

$$\frac{V}{e^{rt}} = A$$

$$A = \frac{V}{e^{rt}}$$

$$A = Ve^{-rt}$$

$e^{-rt}$is also referred as discounting factor.
Reversed roles (like in discrete case) of ***A*** and ***V.***
From growth to negative-growth.

TOPIC: 153: APPLICATIONS OF CONTINUOUS COMPOUNDING

Suppose, the sum of PKR 5000 is invested in an account earning interest at an annual rate of 9%. What will be the balance after 8 years if interest is compounded continuously?

Formula for continuous compounding: $V = Ae^{rt}$

Here, $A = PKR\,5000$, $r = 9\%\ or\ 0.09$, $t = 8\ years$.
Substituting values:

$$\boldsymbol{V = 5000e^{0.09(8)}}$$

$$\boldsymbol{V = PKR\ 10272.17}$$

Interpretation: Principal will increase from PKR 5000 to PKR 10272.17 during after 8 years, if the annual interest is 9%, and there is continuous compounding of interest.

Find the amount K by which \$1 increases in the course of a year when the interest rate is 8% per year and interest is added: (a) yearly; (b) biannually; (c) continuously.

In this case $r = 8/100 = 0.08$, and we obtain

(a) $K = 1.08$ (b) $K = (1 + 0.08/2)^2 = 1.0816$ (c) $K = e^{0.08} \approx 1.08329$

Lesson 32

USE OF LOGARITHMS IN ECONOMICS

TOPIC 154: LOGARITHMS MEANING AND TYPES

Attributed to a Scottish mathematician John Napier (1550-1617)
Etymology: Greek '*logos*' ratio + '*arithmos'* number.
Inverse of exponentiation.
Helps to find the power of a number result of which is known.

$$\mathbf{10^{x} = 100 \Rightarrow \log_{10} 100 = x}$$
$$\mathbf{10^{2} = 100 \Rightarrow \log_{10} 100 = 2}$$

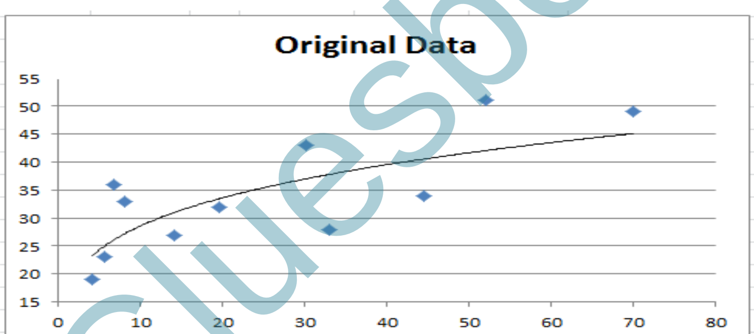
Can have base different than 10:
$\log_5|5| = 1$, $\log_2|2| = 1$
Also 2.718 = e(Euler's number).

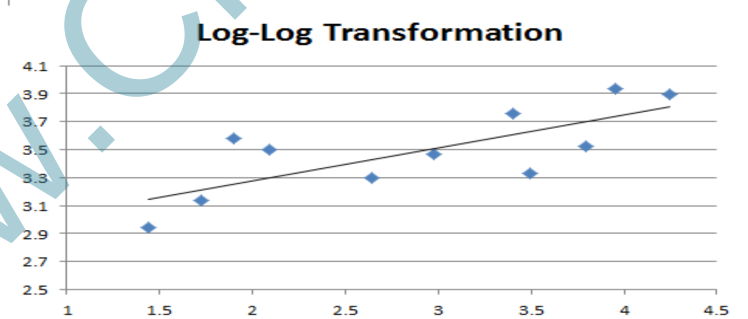
$$\log_e |x| = ln|x|$$

$ln|x|$ is the natural Logarithm of x.
Contrary to natural exponential function $y = e^x$
$y = e^x$ and $y = ln\,|x|$ are inverse functions of each other.

Order independence in inverse.

$$y = ln|e^x| = x$$
$$y = e(ln|x|) = x$$





TOPIC 155: LAWS OF LOGARITHMS

1 - Logarithm of a Product

$$ln(x \cdot y) = ln(x) + ln(y)$$

Or

$$log(x \cdot y) = log(x) + log(y)$$

Example

$\boldsymbol{log(10 \cdot 100) = log(10) + log(100)}$

$\boldsymbol{log(1000) = log(10) + log(100)}$

Using calculator

$3 = 2 + 1$

Economic Instance

$$\boldsymbol{ln(R) = ln(P \cdot Q) = ln(P) + ln(Q)}$$

2 - Logarithm of a Quotient

$\boldsymbol{ln\left(\frac{x}{y}\right) = ln(x) - ln(y)}$

OR

$\boldsymbol{log\left(\frac{x}{y}\right) = log(x) - log(y)}$

Example

$\boldsymbol{log\left(\frac{10}{100}\right) = log(10) - log(100)}$

$\boldsymbol{log(0.1) = log(10) - log(100)}$

Using calculator

$\boldsymbol{-1 = 1 - 2}$

Economic Instance

$\boldsymbol{ln(AC) = ln\left(\frac{C}{Q}\right) = ln(C) - ln(Q)}$

3 - Logarithm of a Power

$$\boldsymbol{ln(x^p) = p \cdot ln(x)}$$

OR

$\boldsymbol{log(x^p) = p \cdot log(x)}$

Example

$$\boldsymbol{log(10^2) = 2 \cdot log(10)}$$
$$\boldsymbol{log(100) = 2 \cdot \{log(10)\}}$$

Using calculator

$$\boldsymbol{2 = 2 \cdot \{1\}}$$

Economic Instance

$$\boldsymbol{ln(Q) = ln(K^{\alpha} L^{\beta})}$$
$$\boldsymbol{ln(Q) = ln(K^{\alpha}) + ln(L^{\beta})}$$
$$\boldsymbol{ln(Q) = \alpha \cdot ln(K) + \beta \cdot ln(L)}$$

Log-linearized form of Cobb-Douglas production function.

TOPIC 156: LAWS OF LOGARITHMS FOR TRANSFORMATION OF CES PRODUCTION FUNCTION

Consider a Standard Cobb-Douglas Production Function $Q = AK^{\alpha}L^{\beta}$

Taking logarithm on both sides.

$$\ln|Q| = \ln|AK^{\alpha}L^{\beta}|$$

$$\ln|Q| = \ln|A| + \alpha \ln|K| + \beta \ln|L|$$

- log-linearized form of Cobb-Douglas Production function.
- Now α and β are the output elasticities of K and L respectively

Consider a standard CES production function

CES = Constant Elasticity of Substitution

$$q = A\{\alpha K^{-\beta} + (1-\alpha)L^{-\beta}\}^{-1/\beta} \quad \left[\begin{array}{c} A>0 \\ 0<\alpha<1 \\ -1<\beta<\infty \\ \beta\neq 0 \end{array}\right]$$

To linearize we use logarithms

$$\ln|q| = \ln\left|A\{\alpha K^{-\beta} + (1-\alpha)L^{-\beta}\}^{-1/\beta}\right|$$

$$= \ln|A| + \ln\left|\{\alpha K^{-\beta} + (1-\alpha)L^{-\beta}\}^{-1/\beta}\right|$$

$$= \ln|A| + \left(-\frac{1}{\beta}\right) \ln\left|\{\alpha K^{-\beta} + (1-\alpha)L^{-\beta}\}\right|$$

$$\ln|q| = \ln|A| - \frac{1}{\beta} \ln\left[\alpha K^{-\beta} + (1-\alpha)L^{-\beta}\right]$$

Lesson 33

RULES OF DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

TOPIC 157: RULES OF DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Possibility of exponential functions in differentiation.

$$y = e^{f(x)}$$
$$\frac{dy}{dx} = f'(x) \times e^{f(x)}$$

Example: $y = e^{rx}$
Here, $f(x) = rx$
$\Longrightarrow f'(x) = r$
Deploying the rule:

$$\frac{d\{e^{f(x)}\}}{dx} = f'(x) \times e^{f(x)}$$
$$\frac{dy}{dx} = r \times e^{rx} = re^{rx}$$

Possibility of logarithmic functions in differentiation.
$y = ln|f(x)|$ Where $f(x) \not< 0$

$$\frac{dy}{dx} = \frac{d\{ln|f(x)|\}}{dx} = \frac{f'(x)}{f(x)}$$

Example: $y = ln|ax|$
Here $f(x) = ax$
$\Longrightarrow f'(x) = a$
Deploying the rule:

$$\frac{d\{ln|f(x)|\}}{dx} = \frac{f'(x)}{f(x)}$$
$$\frac{d\{ln|ax|\}}{dx} = \frac{a}{ax} = \frac{1}{x}$$

TOPIC 158: OPTIMAL TIMING: A PROBLEM OF WINE STORAGE

Sell wine today ($t = 0$) for $ K or store and sell at higher price.
Higher value (V):.

$$V(t) = Ke^{\sqrt{t}} = K.exp(\sqrt{t}) = K.exp(t)^{1/2}$$

Sell now, ($t = 0$) implies:
$V(0) = K.exp(0)^{1/2} = K.exp(0) = K.(1) = K$

Assumptions

- Storage costs = 0 (container)
- Sunk costs as wine is possessed in prior and not being produced currently.

Therefore:
$\pi = R - C$ reduces to $\pi = R$ as $C = 0$.
As $A(t) = Ve^{-rt}$, value of V is known.

$$V(t) = Ke^{\sqrt{t}}$$
$$A(t) = Ke^{\sqrt{t}}e^{-rt}$$

$A(t) = Ke^{\sqrt{t}-rt}$ $[A(t^*)_{max}: t^* = \ ?]$

$$\boldsymbol{ln|A(t)| = ln|K| + ln|e^{\sqrt{t}-rt}|}$$

$$\frac{dln|A(t)|}{dt} = \frac{d}{dt}(ln|K| + \sqrt{t} - rt)$$

$$\frac{1}{A}\frac{d|A(t)|}{dx} = \frac{d}{dt}(ln|K|) + \frac{d}{dt}(\sqrt{t}) - \frac{d}{dt}(rt)$$

$$\frac{1}{A}\frac{d|A(t)|}{dx} = \frac{d}{dt}(ln|K|) + \frac{d}{dt}(\sqrt{t}) - \frac{d}{dt}(rt)$$

$$\frac{1}{A}\frac{d|A(t)|}{dx} = \frac{1}{2\sqrt{t}} - r$$

$$\frac{d|A(t)|}{dx} = A\left(\frac{1}{2\sqrt{t}} - r\right)$$

F.o.C implies: $\frac{d|A(t)|}{dx} = 0$

$$A\left(\frac{1}{2\sqrt{t}} - r\right) = 0$$

$r = \frac{1}{2\sqrt{t}}$ OR $t^* = \frac{1}{4r^2}$

If $r = 10\% = 0.1$,

Then $(t^*)_{r=10\%} = 25\ years$

Storage should be done for 25 years before selling.

$$V(t) = Ke^{\sqrt{t}}$$

For Rate of Growth of V

$$\frac{d}{dt}\{V(t)\} = \frac{d}{dt}Ke^{\sqrt{t}}$$

$$= K\frac{1}{2\sqrt{t}}e^{\sqrt{t}}$$

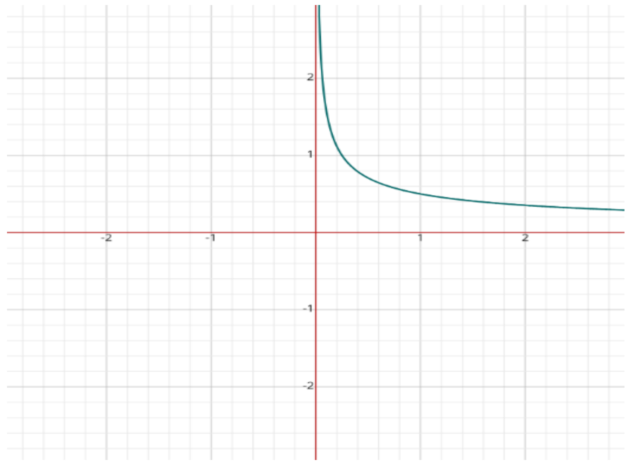
$$G_{V(t)} = \frac{K\frac{1}{2\sqrt{t}}e^{\sqrt{t}}}{Ke^{\sqrt{t}}}$$

$$G_{V(t)} = \frac{1}{2\sqrt{t}} = r$$

Second Order Condition

$$\frac{d}{dt}\left\{A\left(-r+\frac{1}{2\sqrt{t}}\right)\right\} = -\frac{A}{4t^{\frac{3}{2}}}$$

$\frac{d^2}{dt^2}(A(t)) < 0$ implies $A(t)_{max}$



TOPIC 159: OPTIMAL TIMING: A PROBLEM OF TIMBER CUTTING

Value of timber already planted in $1000.

$$V(t) = 2^{\sqrt{t}}$$

Harvest now, $(t = 0)$ implies:

$$V(0) = 2^{(0)^{1/2}} = 1$$

$V_{max}(t^*)$: $t^* = ?$

Assumptions

- Upkeep costs = 0
- Sunk costs as plants are already grown not being grown currently.

Therefore:
$\pi = R - C$ reduces to $\pi = R$ as $C = 0$.
As $A(t) = Ve^{-rt}$, value of V is known.

$$V(t) = 2^{\sqrt{t}}$$
$$A(t) = 2^{\sqrt{t}}e^{-rt}$$
$$ln|A| = ln\left|2^{\sqrt{t}}\right| + ln|e^{-rt}|$$
$$= \sqrt{t}.ln|2| + (-rt)ln|e|$$
$$= \sqrt{t}.ln|2| - rt$$

First Order Condition

$$\frac{\left(dA/_{dt}\right)}{A} = ln|2|.\frac{1}{2\sqrt{t}} - r$$
$$\frac{dA}{dt} = A.\left(ln|2|.\frac{1}{2\sqrt{t}} - r\right) = 0$$
$$A.\left(ln|2|.\frac{1}{2\sqrt{t}} - r\right) = 0$$
$$ln|2|.\frac{1}{2\sqrt{t}} - r = 0$$

$\frac{ln|2|}{2\sqrt{t}} = r$ Or $t^* = \left(\frac{ln|2|}{2}\right)^2$

t^*:Optimum number of years of growth.
Greater the discount rate, earlier should the timber be cut.
Numerically: if $r = 5\%$

$t^* = 48\ yrs,$

$A.\left(ln|2|.\frac{1}{2\sqrt{t}} - r\right) = 0$

$ln|2|.\frac{1}{2\sqrt{t}} - r = 0$

$\frac{ln|2|}{2\sqrt{t}} = r$ Or $t^* = \left(\frac{ln|2|}{2}\right)^2$

Present value at $t^*: A(t^*)$:

$$A(t^*) = 2^{\sqrt{t^*}}e^{-rt^*}$$

$$A(48) = 2^{\sqrt{48}}e^{-0.05.(48)}$$

$A(48) = \$11.0674$ (thousands)

$$\frac{1}{A}\frac{d|A(t)|}{dx} = \frac{d}{dt}(ln|K|) + \frac{d}{dt}(\sqrt{t}) - \frac{d}{dt}(rt)$$

$$\frac{1}{A}\frac{d|A(t)|}{dx} = \frac{1}{2\sqrt{t}} - r$$

$$\frac{d|A(t)|}{dx} = A\left(\frac{1}{2\sqrt{t}} - r\right)$$

$$\frac{d|A(t)|}{dx} = 0$$

$$A\left(\frac{1}{2\sqrt{t}} - r\right) = 0$$

$r = \frac{1}{2\sqrt{t}}$ Or $t^* = \frac{1}{4r^2}$

If $r = 10\% = 0.1,$

Then $(t^*)_{r=10\%} = 25\ years$

Storage should be done for 25 years before selling.

$$V(t) = Ke^{\sqrt{t}}$$

For Rate of Growth of V

$$\frac{d}{dt}\{V(t)\} = \frac{d}{dt}Ke^{\sqrt{t}}$$

$$= K\frac{1}{2\sqrt{t}}e^{\sqrt{t}}$$

$$G_{V(t)} = \frac{K\frac{1}{2\sqrt{t}}e^{\sqrt{t}}}{Ke^{\sqrt{t}}}$$

$$G_{V(t)} = \frac{1}{2\sqrt{t}}$$

Second Order Condition

$$\frac{d}{dt}\left\{A\left(-r+\frac{1}{2\sqrt{t}}\right)\right\} = -\frac{A}{4t^{\frac{3}{2}}}$$

$\frac{d^2}{dt^2}(A(t)) < 0$ implies maximum

TOPIC 160: OPTIMAL TIMING: LAND PURCHASE FOR SPECULATION

Land purchased for speculation.
Value is increasing:

$$V(t) = 1000e^{\sqrt[3]{t}}$$

Sell now, $(t = 0)$ implies:

$$V(0) = 1000.e^{(0)^{1/3}} = 1000$$

$$V_{max}(t^*): t^* = ?$$

Assumptions

- Upkeep Costs = 0
- Sunk Costs as plants are already grown not being grown currently.

As $A(t) = Ve^{-rt}$, value of V is known.

$$V(t) = 1000.e^{\sqrt[3]{t}}$$

$$A(t) = 1000.e^{\sqrt[3]{t}}e^{-rt} = 1000.e^{\sqrt[3]{t}-rt}$$

$$ln|A| = ln|1000| + ln\left|e^{\sqrt[3]{t}-rt}\right|$$

$$= ln|1000| + \sqrt[3]{t} - rt$$

Given that $r = 9\%$

$$ln|A| = ln|1000| + \sqrt[3]{t} - 0.09t$$

$$\frac{\left(dA/_{dt}\right)}{A} = \frac{1}{3t^{2/3}} - 0.09$$

$$\frac{dA}{dt} = \left(\frac{1}{3t^{2/3}} - 0.09\right).A$$

First Order Condition

$$\frac{dA}{dt} = \left(\frac{1}{3t^{2/3}} - 0.09\right).A = 0$$

$$\left(\frac{1}{3t^{2/3}} - 0.09\right).A = 0$$

$$\frac{1}{3t^{2/3}} - 0.09 = 0$$

$$\frac{1}{3t^{2/3}} = 0.09 \Rightarrow \frac{1}{0.09} = 3t^{2/3}$$

$$\frac{1}{0.27} = t^{2/3} \Rightarrow t = \left(\frac{1}{0.27}\right)^{3/2}$$

$$t^* = 7.127\ years$$

t^*:Optimum number of years.
Land should be held for 7.127 years before sale.

Second Order Condition

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = \frac{d}{dt}\left\{\left(\frac{1}{3t^{2/3}} - 0.09\right).A\right\}$$

$$\frac{d^2A}{dt^2} = = \frac{d}{dt}\left\{\left(\frac{1}{3}t^{-2/3} - 0.09\right).A\right\}$$

$$= \left\{\left(-\frac{2}{9}t^{-5/3}\right).A\right\} + \left\{\left(\frac{1}{3}t^{-2/3} - 0.09\right).\frac{dP}{dt}\right\}$$

$= \frac{-2P}{9\sqrt[3]{t^5}}$

$\frac{d^2}{dt^2}(A(t)) < 0$ implies maximum

TOPIC 161: OPTIMAL TIMING: ART COLLECTION

Estimated value of art collection of a deceased painter:

$$V(t) = 200000(1.25)^{\sqrt[3]{t^2}}$$

Sell now, $(t = 0)$ implies:

$$V(0) = V(t) = 200000(1.25)^{\sqrt[3]{0^2}} = 200000$$

$V_{max}(t^*)$: $t^* = ?$

As $A(t) = Ve^{-rt}$, value of V is known.

$$V(t) = 200000.(1.25)^{\sqrt[3]{t^2}}$$

$$A(t) = 200000.(1.25)^{\sqrt[3]{t^2}}e^{-rt}$$

$$A(t) = 200000.(1.25)^{t^{2/3}}e^{-rt}$$

$$ln|A| = ln|200000| + ln\left|1.25^{t^{2/3}}\right| + ln|e^{-rt}|$$

$$ln|A| = ln|200000| + t^{2/3}.ln|1.25| - rt$$

Given that $r = 6\%$

$$ln|A| = ln|200000| + t^{2/3}$$
$$ln|1.25| - 0.06t$$

$$\frac{\left(dA/_{dt}\right)}{A} = \frac{2}{3}ln|1.25|\left(t^{-1/3}\right) - 0.06$$

First Order Condition

$$dA/_{dt} = \left\{\frac{2}{3}ln|1.25|\left(t^{-1/3}\right) - 0.06\right\}A$$

$$dA/_{dt} = \left\{\frac{2}{3}ln|1.25|\left(t^{-1/3}\right) - 0.06\right\}A = 0$$

$$\frac{2}{3}ln|1.25|\left(t^{-1/3}\right) - 0.06 = 0$$

$$\frac{2}{3}ln|1.25|\left(t^{-1/3}\right) = 0.06$$

$$t^{-1/3} = \frac{3(0.06)}{2.ln|1.25|} \Rightarrow t = \left\{\frac{3(0.06)}{2.ln|1.25|}\right\}^{-3}$$

$t^* = \left\{\frac{2.ln|1.25|}{3(0.06)}\right\}^3 = 15.24$ years

Art collection should be sold after 15.24 years.

TOPIC 162: OPTIMAL TIMING: DIAMOND PURCHASE

Estimated value of diamond, bought for investment purpose, is:

$$V(t) = 250000(1.75)^{\sqrt[4]{t}}$$

Sell now, $(t = 0)$ implies:

$$V(0) = V(t) = 250000(1.45)^{\sqrt[4]{0}} = 250000$$

$V_{max}(t^*)$: $t^* = ?$

As $A(t) = Ve^{-rt}$, value of V is known.

$$V(t) = 250000.(1.75)^{\sqrt[4]{t}}$$

$$A(t) = 250000.(1.75)^{\sqrt[4]{t}}e^{-rt}$$

$$A(t) = 250000.(1.75)^{t^{1/4}}e^{-rt}$$

$$ln|A| = ln|250000| + ln\left|1.75^{t^{1/4}}\right| + ln|e^{-rt}|$$

$$ln|A| = ln|250000| + t^{1/4}.ln|1.75| - rt$$

Given that $r = 7\%$

$$ln|A| = ln|250000| + t^{1/4}$$

$$ln|1.75| - 0.07t$$

$$\frac{\left(dA/_{dt}\right)}{A} = \frac{1}{4}ln|1.75|\left(t^{-3/4}\right) - 0.07$$

First Order Condition

$$dA/_{dt} = \left\{\frac{1}{4}ln|1.75|\left(t^{-3/4}\right) - 0.07\right\}A$$

$$dA/_{dt} = \left\{\frac{1}{4}ln|1.75|\left(t^{-3/4}\right) - 0.07\right\}A = 0$$

$$\frac{1}{4}ln|1.75|\left(t^{-3/4}\right) - 0.07 = 0$$

$$\frac{1}{4}ln|1.75|\left(t^{-3/4}\right) = 0.07$$

$$t^{-3/4} = \frac{4(0.07)}{ln|1.75|} \Rightarrow t = \left\{\frac{4(0.07)}{ln|1.75|}\right\}^{-4/3}$$

$$t^* = \left\{\frac{ln|1.75|}{4(0.07)}\right\}^{4/3} = 2.52 \text{ years}$$

Diamond should be sold after 2.52 years.

Lesson 34

FINDING THE RATE OF GROWTH USING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

TOPIC 163: FINDING THE RATE OF GROWTH USING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Assume $y = f(t)$
y depends on t (time).
Instantaneous Growth Rate:
$r_y = \frac{\left(\frac{dy}{dt}\right)}{y} = \frac{f'(t)}{f(t)} = \frac{Marginal\ Function}{Total\ Function} = \frac{d}{dt}\{ln|f(t)|\}$
Example: $V = Ae^{rt}$

Growth Rate of V.

$$ln|V| = ln|A| + rt.ln|e|$$
$$= ln|A| + rt.1$$
$$= ln|A| + rt$$

Taking derivative w.r.t t.

$$\frac{d}{dt}ln|V| = \frac{d}{dt}\{ln|A| + rt\}$$
$= \frac{d}{dt}\{ln|A|\} + \frac{d}{dt}\{rt\} = 0 + r = r$ **(Rate of growth of V is r)**

TOPIC 164: GROWTH OF EXPORTS OF A COUNTRY

A country export both goods (G) and services (S).
Both depend on time i.e. $G(t)$ and $S(t)$.
Total exports: $X(t) = G(t) + S(t)$.
Rate of growth of exports $r_X = \frac{X'(t)}{X}$

$$X'(t) = \frac{d}{dt}\{X(t)\}$$
$$= \frac{d}{dt}\{G(t) + S(t)\}$$
$$= \frac{d}{dt}\{G(t)\} + \frac{d}{dt}\{S(t)\}$$
$$X'(t) = G'(t) + S'(t)$$

Substituting in $r_X = \frac{X'(t)}{X}$

$$r_X = \frac{G'(t) + S'(t)}{X}$$

$r_X = \frac{G'(t)}{X} + \frac{S'(t)}{X}$
$= \frac{G}{G}\frac{G'(t)}{X} + \frac{S}{S}\frac{S'(t)}{X}$
$= \frac{G}{X}\frac{G'(t)}{G} + \frac{S}{X}\frac{S'(t)}{S}$
$r_X = \frac{G}{X}r_G + \frac{S}{X}r_S$ …………(A)

Given $r_G = \frac{a}{t}$ and $r_S = \frac{b}{t}$

Equation (A) becomes:

$r_X = \frac{G}{X}\left(\frac{a}{t}\right)+\frac{S}{X}\left(\frac{b}{t}\right)$

$$r_X = \frac{G \cdot a + S \cdot b}{X \cdot t}$$

TOPIC 165: FINDING THE POINT ELASTICITY

Using the usual formula of price elasticity of demand:

$$\epsilon_d = \left(\frac{dQ}{dP}\right) / \left(\frac{Q}{P}\right)$$

$$\epsilon_d = \left(\frac{dQ}{dP}\right)\left(\frac{P}{Q}\right)$$

$$\epsilon_d = \left(\frac{dQ}{Q}\right)\left(\frac{P}{dP}\right)$$

$$\epsilon_d = \frac{\left(\frac{dQ}{Q}\right)}{\left(\frac{dP}{P}\right)}$$

$$\epsilon_d = \frac{d(ln|Q|)}{d(ln|P|)}$$

Assuming a specific demand function:

$$Q = k/P$$

$$ln|Q| = ln|k| - ln|P|$$

Logarithmically differentiating w.r.t P.

$$\frac{d}{d(ln|P|)}(ln|Q|) = \frac{d}{d(ln|P|)}(ln|k|) - \frac{d}{d(ln|P|)}(ln|P|)$$

$= 0 - 1$

$$\epsilon_d = -1$$

$|\epsilon_d| = 1$ (Unitary elastic demand curve)

TOPIC 166: RATES OF GROWTH OF POPULATION, CONSUMPTION, AND PER CAPITA CONSUMPTION

Growth rate of consumption (C) is α.

Growth rate of population (H) is β.

Consumption per capita (P).

$P = \left(\frac{C}{H}\right)$

$ln|P| = ln\left|\frac{C}{H}\right|$

$ln|P| = ln|C| - ln|H|$

$\frac{d}{dt} ln|P(t)| = \frac{d}{dt} ln|C(t)| - \frac{d}{dt} ln|H(t)|$

$$\frac{P'(t)}{P(t)} = \frac{C'(t)}{C(t)} - \frac{H'(t)}{H(t)}$$

$$r_P = r_C - r_H$$

Given $r_C = \alpha$ and $r_H = \beta$

$$r_P = \alpha - \beta$$

Rate of growth of a quotient variable is equal to difference of individual growth rates.

TOPIC 167: RATE OF GROWTH OF PER CAPITA EMPLOYMENT

Growth rate of employment opportunities (E) is a.
Growth rate of population (P) is b.
Employment per capita (C).

$C = \left(\frac{E}{P}\right)$

$ln|C| = ln\left|\frac{E}{P}\right|$

$ln|C| = ln|E| - ln|P|$

$\frac{d}{dt}ln|C(t)| = \frac{d}{dt}ln|E(t)| - \frac{d}{dt}ln|P(t)|$

$$\frac{C'(t)}{C(t)} = \frac{E'(t)}{E(t)} - \frac{P'(t)}{P(t)}$$

$$r_C = r_E - r_P$$

Given $r_E = a$ and $r_P = b$

$$r_C = a - b$$

If $r_E = 4\%$ and $r_P = 2.5\%$

$r_C = 4\% - 2.5\% = 1.5\%$

Growth rate of a per capita employment is equal to difference of growth rate of employment and population, respectively.

TOPIC 168: RATE OF GROWTH OF EXPORT EARNINGS OF A COUNTRY

Two exports of a country **C** and **B.**

$C = C(t_o) = 4$, and $B = B(t_o) = 1$.

C grows at 10% and B grows at 20%.

$$E = C + B$$

$$ln|E| = ln|C + B|$$

Differentiate w.r.t.

$$\frac{d(ln|E|)}{dt} = \frac{d(ln|C + B|)}{dt}$$

$$G_E = \frac{C'(t) + B'(t)}{C + B}$$

$G_C = \frac{C'(t)}{C}$ and $G_B = \frac{B'(t)}{B}$

Imply:

$C.G_C = C'(t)$and $B.G_B = B'(t)$

$\frac{C'(t)+B'(t)}{C+B} = \frac{C.G_C+B.G_B}{C+B}$

$= \left(\frac{C}{C+B}\right)G_C + \left(\frac{B}{C+B}\right)G_B$

Given $G_C = 10\%$ and $G_B = 20\%$.

$= \left(\frac{4}{4+1}\right)0.1 + \left(\frac{1}{4+1}\right)0.2$

$$G_E = 12\%$$

TOPIC 169: RATE OF GROWTH OF SALES

Sales function is:

$$S(t) = 100000 \cdot e^{0.5\sqrt{t}}$$

$$ln|S(t)| = ln|100000| + ln|e^{0.5\sqrt{t}}|$$
$$ln|S(t)| = ln|100000| + 0.5\sqrt{t}$$
$$\frac{1}{S}\cdot\frac{dS}{dt} = \frac{0.25}{\sqrt{t}}$$
$$\frac{dS}{dt} = \frac{0.25\cdot S}{\sqrt{t}}$$
$$G_S = \frac{\left(\frac{dS}{dt}\right)}{S} = \frac{\left(\frac{0.25\cdot S}{\sqrt{t}}\right)}{S}$$
$$G_S = \frac{0.25}{\sqrt{t}}$$

Sales function is:

$$G_S = \frac{0.25}{\sqrt{t}}$$

Assuming $t = 4$,

$$G_S = \frac{0.25}{4} = 0.125 = 12.5\%$$

Sales shall grow at 12.5% if 4 years are allowed.

TOPIC 170: RATE OF GROWTH OF PROFIT

In addition to maximization of profit, rate of growth of profit can also be of interest for a firm.
Profit function is:

$$\pi(t) = 250000\cdot e^{1.2\sqrt[3]{t}}$$

$$\pi(t) = 250000\cdot e^{1.2\cdot t^{\frac{1}{3}}}$$

$$ln|\pi(t)| = ln|250000| + ln|e^{1.2\cdot t^{\frac{1}{3}}}|$$

$$ln|\pi(t)| = ln|250000| + 1.2\cdot t^{\frac{1}{3}}$$

$$\frac{1}{\pi}\cdot\frac{d\pi}{dt} = \frac{0.4}{t^{\frac{2}{3}}} \Rightarrow \frac{d\pi}{dt} = \frac{0.4\cdot\pi}{t^{\frac{2}{3}}}$$

$$G_\pi = \frac{\left(\frac{d\pi}{dt}\right)}{\pi} = \frac{\left(\frac{0.4\cdot\pi}{t^{\frac{2}{3}}}\right)}{\pi} = \frac{0.4}{t^{\frac{2}{3}}}$$

Growth rate of profit function is:

$$G_\pi = \frac{0.4}{t^{\frac{2}{3}}}$$

Assuming $t = 8$,

$$G_\pi = \frac{0.4}{8^{\frac{2}{3}}} = 0.1 = 10\,\%$$

Profit shall grow at 10% if 8 years are allowed.

Lesson 35

CONCEPT OF OPTIMIZATION

TOPIC 171: CONCEPT OF OPTIMIZATION

Etymology optimum: Latin *'Optimus'* 'best'

Process of maximizing favorable variables and minimizing unfavorable variables.

Assume a function:

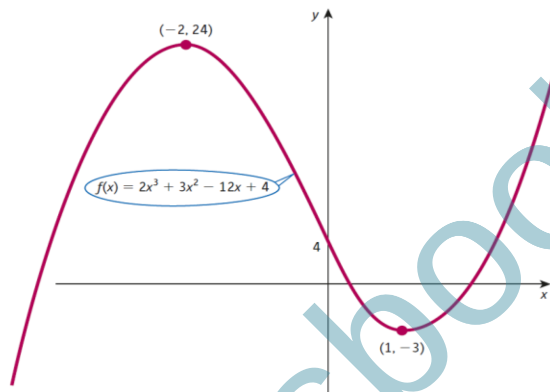
$$y = f(x)$$

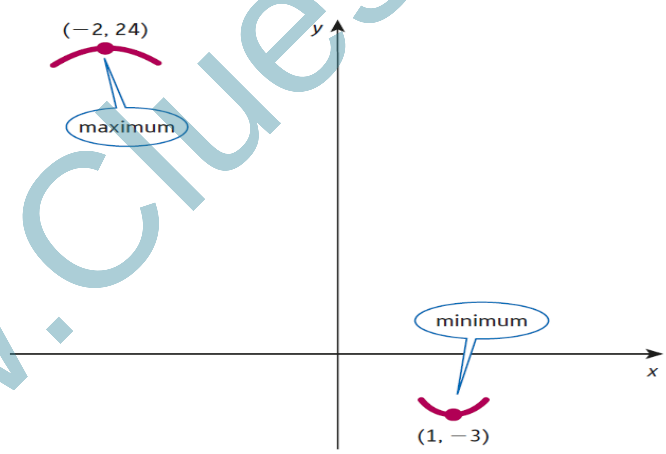
In cubic specification:

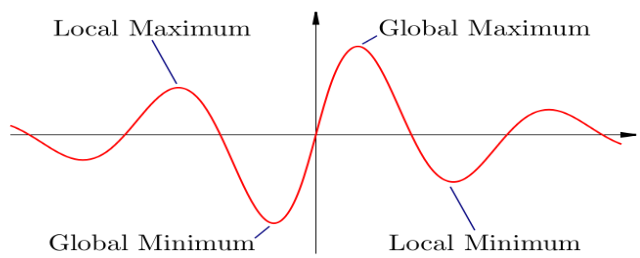
$y = f(x) = 2x^3 + 3x^2 - 12x + 4$

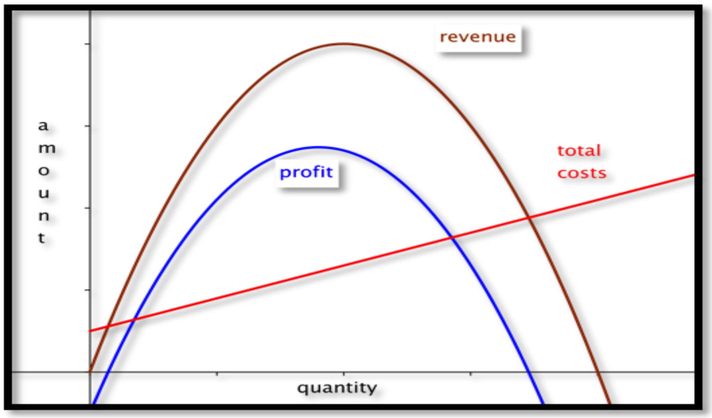
Two wiggles.

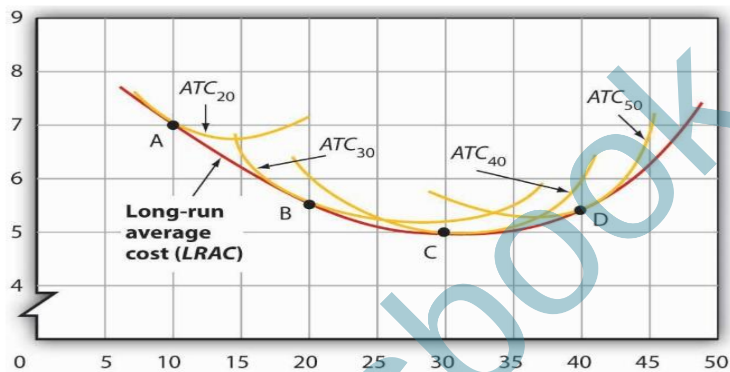
One upwards and other downwards.











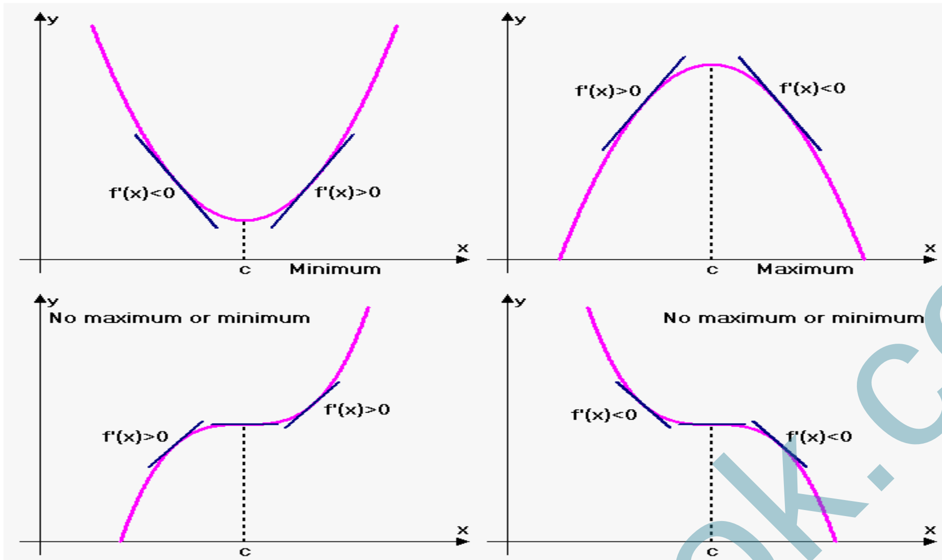
Extremum vs Extrema
Maximum vs Maxima
Minimum vs Minima

TOPIC 172: CALCULUS APPROACH TO OPTIMIZATION: 1ST ORDER TEST

Considering a function $y = f(x)$.
Which is continuous and can have extremum/extrema.
Finding the critical value at which function is optimized.

Since slope at maximum or a minimum is zero.
Slope = $\frac{dy}{dx} = f'(x) = 0$.
a.k.a First order condition (F.o.C).

a.k.a Necessary condition for optimization.



Let c be a critical value of f.

$f(x)$ left of c	$f(x)$ right of c	$f(c)$
Decreasing	Increasing	local minimum at c
Increasing	Decreasing	local maximum at c
Increasing	Increasing	not an extremum
Decreasing	Decreasing	not an extremum

Instance:

$$y = 2x^3 + 3x^2 - 12x + 4$$

Critical Points of $2x^3 + 3x^2 - 12x + 4$

$$f'(x) = 6x^2 + 6x - 12$$

$$6x^2 + 6x - 12 = 0$$

Solve with the quadratic formula

$$x = \frac{-6 + \sqrt{6^2 - 4 \cdot 6(-12)}}{2 \cdot 6} : \quad 1$$

$$x = \frac{-6 - \sqrt{6^2 - 4 \cdot 6(-12)}}{2 \cdot 6} : \quad -2$$

$$x = -2, x = 1$$

For $y = 2x^3 + 3x^2 - 12x + 4$

Subsititute $x = -2$

$$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 4$$

$$2(-2)^3 + 3(-2)^2 - 12(-2) + 4 = 24$$

$$y = 24$$

$$x = -2, y = 24$$

For $y = 2x^3 + 3x^2 - 12x + 4$

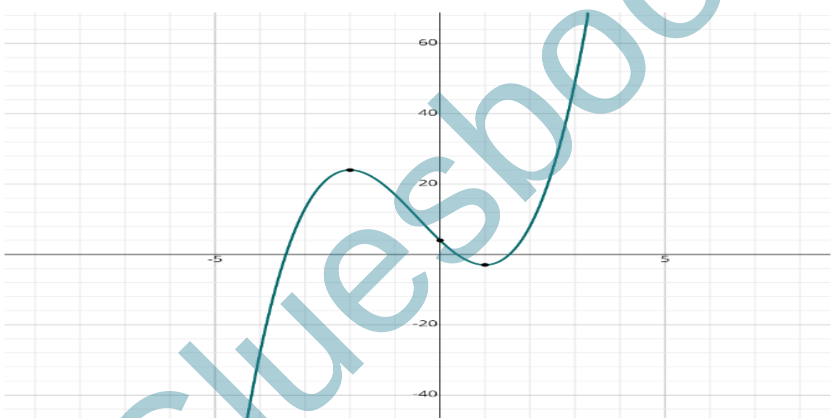
Subsititute $x = 1$

$$y = 2 \cdot 1^3 + 3 \cdot 1^2 - 12 \cdot 1 + 4$$

$$2 \cdot 1^3 + 3 \cdot 1^2 - 12 \cdot 1 + 4 = -3$$

$$y = -3$$

$$x = 1, y = -3$$



TOPIC 173: AVERAGE COST ANALYSIS

Consider an Average Cost Function.

$$AC(Q) = Q^2 - 5Q + 8$$

For Optimization, one need first order and second order conditions.

Taking derivative. w.r.t Q.

$$\frac{d\{AC(Q)\}}{dQ} = \frac{d}{dQ}\{Q^2 - 5Q + 8\}$$

$$AC'(Q) = 2Q - 5 \gtrless 0$$ [First order Condition $f'(x) = 0$]

Assuming $AC'(Q) = 0$

$$2Q - 5 = 0$$

$$Q^* = 5/2 = 2.5$$ [Critical value of Q]

Taking derivative w.r.t Q again

$$\frac{d^2\{AC(Q)\}}{dQ^2} = \frac{d}{dQ}\{2Q-5\}$$

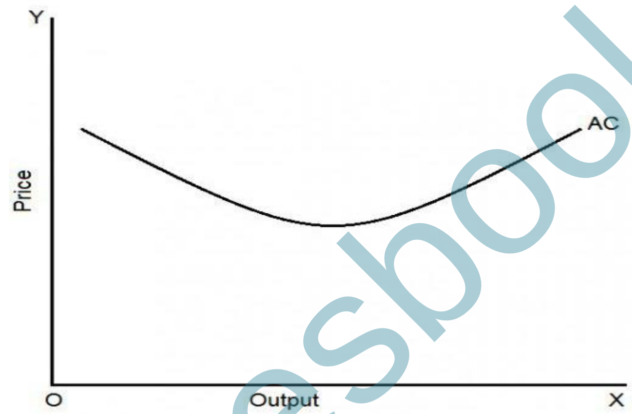
$$AC''(Q) = 2 > 0 \Rightarrow \text{Minimum at } Q^* = 2.5$$

Testing the results at $Q = 2.4, 2.5, 2.6$

$$(AC)_{Q=2.4} = (2.4)^2 - 5(2.4) + 8 = 1.76$$

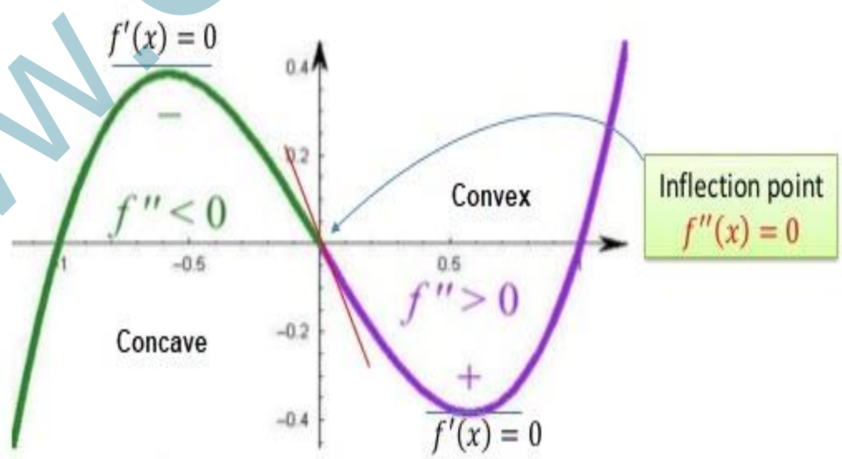
$$(AC)_{Q=2.5} = (2.5)^2 - 5(2.5) + 8 = 1.75 \quad \text{Minimum}$$

$$(AC)_{Q=2.6} = (2.6)^2 - 5(2.6) + 8 = 1.76$$

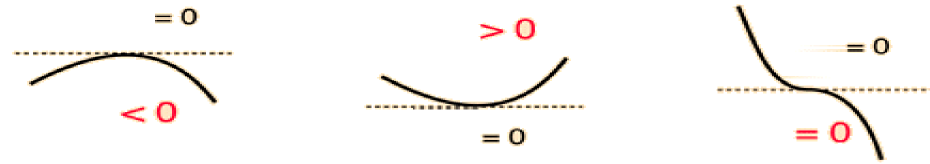


TOPIC 174: CALCULUS APPROACH TO OPTIMIZATION: 2ND ORDER TEST

Considering a function $y = f(x)$.
Which is continuous and can have extremum/extrema.
Having found the critical value(s) at which function is optimized, one needs to confirm if function is actually maximized or minimized or just a 'bend in curve' (inflection).



The second derivative demonstrates whether a point with zero first derivative is a maximum, a minimum, or an inflexion point.



For a maximum, the second derivative is negative. The slope of the curve (first derivative) is at first positive, then goes through zero to become negative.

For a minimum, the second derivative is positive. The slope of the curve = first derivative is at first negative, then goes through zero to become positive.

For an inflexion point, the second derivative is zero at the same time the first derivative is zero. It represents a point where the curvature is changing its sense. Inflexion points are relatively rare in nature.

Rate of change of slope.

Rate of change of Slope = $\frac{d\left(\frac{dy}{dx}\right)}{dx} = f''(x) \lesseqgtr 0$.

a.k.a Second order condition (S.o.C)

a.k.a Sufficient condition for optimization

Instance:

$$y = 2x^3 + 3x^2 - 12x + 4$$
$$x^* = -2,\ 1$$
$$f''(x) = 12x + 6$$
$$f''(x^*) = 12x^* + 6$$

For $x^* = -2$

$$f''(-2) = 12(-2) + 6 = -18 < 0$$

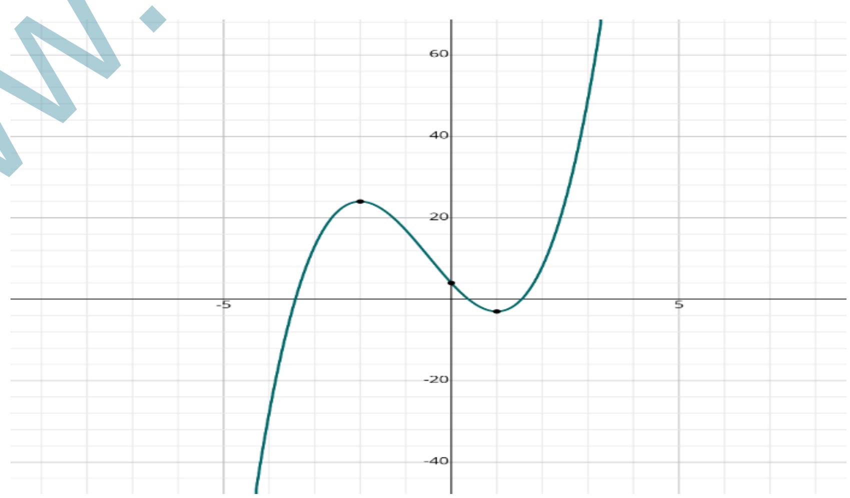
$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 4 = \mathbf{24}$ (maximum at $x^* = -2$)

For $x^* = 1$

$$f''(1) = 12(1) + 6 = 18 > 0$$

$y = 2(1)^3 + 3(1)^2 - 12(1) + 4 = \mathbf{-3}$ (minimum at $x^* = 1$)

For $x = -0.5$, $f''(-0.5) = 0$, $y = f(-0.5) = \mathbf{10.5}$ (inflection)



TOPIC 175: MATRIX APPROACH TO OPTIMIZATION: 2ND ORDER TEST – HESSIAN

Hessian $|H|$ is a determinant with all the second-order partial derivatives.
2nd order direct partials on the principal diagonal
2nd order cross partials off the principal diagonal.

$$|\boldsymbol{H}| = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

Where, $F_{xy} = F_{yx}$ (Young's theorem)
Principal minors = 2
1st Principal minor: $|H_1|$

$$|\boldsymbol{H_1}| = |F_{xx}|$$

2nd Principal minor: $|H_2|$

$$|\boldsymbol{H_2}| = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} = |\boldsymbol{H}|$$

If both $|H_1|$ and $|H_2|$ are positive then minimum will be evident.
$|\boldsymbol{H_1}| > 0 \wedge |\boldsymbol{H_2}| > 0 \Rightarrow$ **Minimum**
If $|H_1|$ and $|H_2|$ have alternative signs then minimum will be evident.
$|\boldsymbol{H_1}| < 0, |\boldsymbol{H_2}| > 0 \Rightarrow$ **Maximum**

If $F(x,y) = 3x^2 - xy + 2y^2 - 4x - 7y + 12$

$$F_x = \frac{\partial F(x,y)}{\partial x} = 6x - y - 4$$

$$F_{xx} = \frac{\partial (F_x)}{\partial x} = 6, F_{xy} = \frac{\partial (F_x)}{\partial y} = -1$$

$$F_y = \frac{\partial F(x,y)}{\partial y} = -x + 4y - 7$$

$$F_{yy} = \frac{\partial (F_y)}{\partial y} = 4, F_{yx} = \frac{\partial (F_y)}{\partial x} = -1$$

$$|\boldsymbol{H}| = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix}$$

$$|\boldsymbol{H_1}| = |F_{xx}| = |6| = 6 > 0$$

$$|\boldsymbol{H_2}| = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix} = 23 > 0$$

Since both $|H_1|$ and $|H_2|$ are positive, there exists a minimum.

If $z(x,y) = 2 - x^2 - xy - y^2$, Evaluate 2nd order condition using Hessian Determinant.

For $F(x_1, x_2, x_3)$

$$|\boldsymbol{H}| = \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix}$$

$|H_1| < 0, |H_2| > 0, |H_3| < 0$ implies a maximum.
$|H_1| > 0, |H_2| > 0, |H_3| > 0$ implies a minimum.

N-number of variables can be dealt with using Hessian determinant.

Lesson 36

PROFIT MAXIMIZATION ANALYSIS

TOPIC 176: PROFIT MAXIMIZATION ANALYSIS

Considering Revenue and Cost functions in their general forms:

$$R = f(Q) \quad \& \quad C = f(Q)$$

- Both revenue and costs are expressed in terms of output.
- Both revenue and costs are increasing function of output.
- Profit function can be formulated by taking difference of the two above-mentioned functions:

$$\pi(Q) = R(Q) - C(Q)$$

- Note that π is also in terms of Q.

- First-order condition implies:

$$\frac{d}{dQ}\{\pi(Q)\} = \frac{d}{dQ}\{R(Q) - C(Q)\} = 0$$

$$\pi'(Q) = \frac{d}{dQ}\{R(Q)\} - \frac{d}{dQ}\{C(Q)\}$$

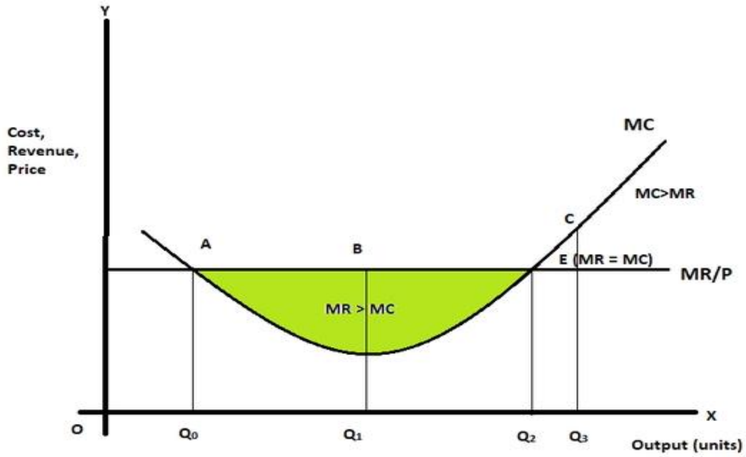
$$\pi'(Q) = R'(Q) - C'(Q) = 0$$

$$R'(Q) - C'(Q) = 0$$

$$R'(Q) = C'(Q)$$

$$\boxed{MR = MC}$$ π maximizing condition.

- $MR = MC$ gives rise to a certain value of Q, which is known as the critical value of Q, (Q^*). Q^* maximizes the profit i.e. ($\pi_{max.}$).



- However, F.o.C may not necessarily lead to a maximum.
- Therefore S.o.C is deployed.

$$\frac{d}{dQ}\left(\frac{d\pi}{dQ}\right) = \frac{d}{dQ}(\pi(Q))$$

$$= \frac{d}{dQ}\{R'(Q) - C'(Q)\}$$

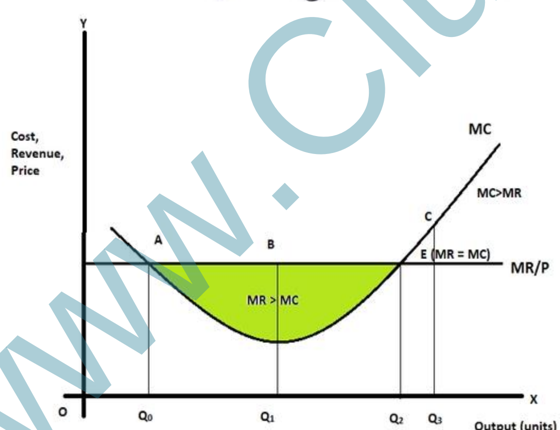
$$= \frac{d}{dQ}\{R'(Q)\} - \frac{d}{dQ}\{C'(Q)\}$$

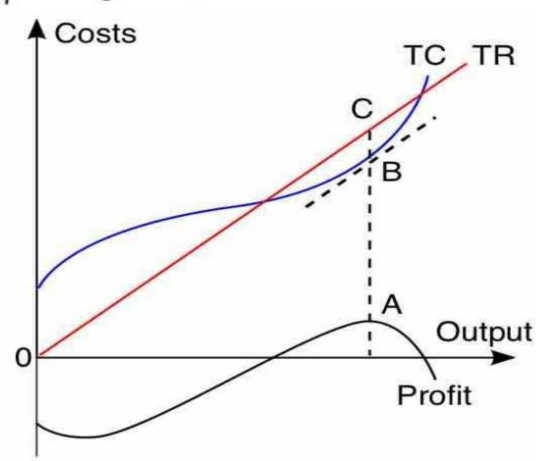
$$= R''(Q) - C''(Q) < 0 \quad [\text{For a maximum}]$$

$$R''(Q) < C''(Q)$$

slope of $R'(Q)$ < slope of $C'(Q)$

slope of MR < slope of MC





TOPIC 177: NUMERICAL EXAMPLE OF PROFIT MAXIMIZATION

Considering R(Q) and C(Q) as the revenue and cost functions:

$$R(Q) = 1200Q - 2Q^2$$

$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$

Forming the profit function:

$$\pi(Q) = R(Q) - C(Q)$$

$$= 1200Q - 2Q^2 - (Q^3 - 61.25Q^2 + 1528.5Q + 2000)$$

$$= 1200Q - 2Q^2 - Q^3 + 61.25Q^2 - 1528.5Q - 2000$$

$$\pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

Forming the first order condition.

$$\frac{d}{dQ}\{\pi(Q)\} = \frac{d}{dQ}(-Q^3 + 59.25Q^2 - 328.5Q - 2000)$$

$$= -3Q^2 + 118.5Q - 328.5 = 0$$

Applying quadratic formula.

$a = -3$, $b = 118.5$, $c = -328.5$.

$$Q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

plugging in the values & simplifying, we get

$$Q = 3 \text{ or } 36.5$$

- Both 3 and 36.5 are critical values of Q and lead to maximum/minimum.
- To ascertain the existence of a maximum, we resort to second order condition.

$$\frac{d}{dQ}\left\{\frac{d\pi(Q)}{dQ}\right\} = \frac{d}{dQ}\left\{-3Q^2 + 118.5Q - 328.5\right\}$$

$$\frac{d^2\pi}{dQ^2} = -6Q + 118.5$$

$Q = 3$	$Q = 36.5$
$\left[\frac{d^2\pi}{dQ^2}\right]_{Q=3} = -6(3) + 118.5$	$\left[\frac{d^2\pi}{dQ^2}\right]_{Q=36.5} = -6(36.5) + 118.5$
$= 100.5 > 0$	$= -100.5 < 0$
minium	maximum

- Maximized profit is also easy to find.

$$\{\pi(Q)\}_{Q=36.5} = -(36.5)^3 + 59.25(36.5)^2 - 328.5(36.5) - 2000$$

$$\pi(Q)_{max} = \$16,318.44$$

- Firm can produce 36.5 units of output to maximize the π.

TOPIC 178: PROFIT MAXIMIZATION OF TECHNICALLY RELATED GOODS

Consider a firm in pure competition that produces two goods. Its revenue and cost functions are:

$$R = 15Q_1 + 18Q_2, \quad C = 2Q_1^2 + 2Q_1Q_2 + 3Q_2^2$$

$$MC_1 = \frac{\partial (C)}{\partial Q_1} = 4Q_1 + 2Q_2$$

$$MC_1 = f(Q_1, Q_2)$$

- As MC_1 (marginal cost of first good is not only dependent on its own output, rather on the production of other good as well. One can consider them Technically related goods.

- Profit function

$$\pi = 15Q_1 + 18Q_2 - 2Q_1^2 - 2Q_1Q_2 - 3Q_2^2$$

$$\pi = -2Q_1^2 - 3Q_2^2 + 15Q_1 + 18Q_2 - 2Q_1Q_2$$

first order conditions.

$$f_1 = \frac{\partial (\pi)}{\partial Q_1} = \frac{\partial}{\partial Q_1}\left(-2Q_1^2 - 3Q_2^2 + 15Q_1 + 18Q_2 - 2Q_1Q_2\right) = 0$$

$$f_1 = -4Q_1 + 15 - 2Q_2 = 0 \quad \text{—— ①}$$

$$f_2 = \frac{\partial (\pi)}{\partial Q_2} = -6Q_2 + 18 - 2Q_1 = 0 \quad \text{—— ②}$$

$$\Rightarrow Q_2 = \frac{15 - 4Q_1}{2} \quad \& \quad Q_1 = -3Q_2 + 9$$

$$Q_1 = -3\left(\frac{15 - 4Q_1}{2}\right) + 9 \Rightarrow \boxed{Q_1^* = 2.7}$$

$$Q_2 = \frac{15 - 4Q_1}{2} = \frac{15 - 4(2.7)}{2}$$

$$\boxed{Q_2^* = 2.1}$$

Second Order condition: f_{11} & $f_{22} < 0$ & $(f_{11})(f_{22}) > (f_{12})^2$ } Max.

$f_1 = -4Q_1 + 15 - 2Q_2$ & $f_2 = -6Q_2 + 18 - 2Q_1$

$$f_{11} = \frac{\partial}{\partial Q_1}(f_1) = \frac{\partial}{\partial Q_1}(-4Q_1 + 15 - 2Q_2) = -4 < 0$$

$$f_{12} = \frac{\partial}{\partial Q_2}(f_1) = \frac{\partial}{\partial Q_2}(-4Q_1 + 15 - 2Q_2) = -2 < 0$$

$$f_{21} = \frac{\partial}{\partial Q_1}(f_2) = \frac{\partial}{\partial Q_1}(-6Q_2 + 18 - 2Q_1) = -2 < 0$$

$$f_{22} = \frac{\partial}{\partial Q_2}(f_2) = \frac{\partial}{\partial Q_2}(-6Q_2 + 18 - 2Q_1) = -6 < 0$$

Verification: $f_{11} = -4$, & $f_{22} = -6$

both are negative i.e.

f_{11} & $f_{22} < 0$

$f_{12} = -2$, $f_{21} = -2$, $\Rightarrow f_{12} = f_{21}$ (Young's theorem).

$= (f_{11})(f_{22}) > (f_{12})^2$

$= (-4)(-6) > (-2)^2$

$= 24 > 4$

Therefore π is maximized at $Q_1^* = 2.7$, $Q_2^* = 2.1$

$\pi_{max} = 39.15$

TOPIC 179: PROFIT MAXIMIZATION OF MONOPOLISTIC FIRM PRODUCING RELATED GOODS

Considering a monopolistic competitive firm, that produces two goods that related.

- The inverse demand functions of good 1 and good 2 are:

$$P_1 = 80 - 5Q_1 - 2Q_2$$

$$P_2 = 50 - Q_1 - 3Q_2$$

- Cost function is:

$$C = 3Q_1^2 + Q_1 Q_2 + 2Q_2^2$$

- Profit function $\pi = R_1 + R_2 + C$

$$\pi = P_1 Q_1 + P_2 Q_2 + C$$

$$\pi = (80 - 5Q_1 - 2Q_2)Q_1 + (50 - Q_1 - 3Q_2)Q_2 + (3Q_1^2 + Q_1 Q_2 + 2Q_2^2)$$

$$\pi = 80Q_1 + 50Q_2 - 4Q_1 Q_2 - 8Q_1^2 - 5Q_2^2$$

- First Order Conditions:

$$\pi_1 = \frac{\partial(\pi)}{\partial Q_1} = \frac{\partial}{\partial Q_1}\left(80Q_1 + 50Q_2 - 4Q_1 Q_2 - 8Q_1^2 - 5Q_2^2\right)$$

$$\pi_1 = 80 - 4Q_2 - 16Q_1 = 0 \longleftarrow (1)$$

$$\pi_2 = 50 - 4Q_1 - 10Q_2 = 0 \longleftarrow (2)$$

eq. (1) $\Rightarrow 4Q_2 = 80 - 16Q_1$

$$\boxed{Q_2 = 20 - 4Q_1}$$

$$50 - 4Q_1 - 10(20 - 4Q_1) = 0$$

$$50 - 4Q_1 - 200 + 40Q_1 = 0$$

$$-150 = -36Q_1$$

$$Q_1 = 150/36$$

$$\boxed{Q_1^* = 4.17}$$

$$Q_2^* = 20 - 4(4.17)$$

$$\boxed{Q_2^* = 3.33}$$

Second Order Conditions:

$$\pi_{11},\ \pi_{22} < 0 \ \ \& \ \ (\pi_{11})(\pi_{22}) > (\pi_{12})^2$$

$$\pi_{11} = \frac{\partial}{\partial Q_1}(\pi_1) = \frac{\partial}{\partial Q_1}(80 - 4Q_2 - 16Q_1) = -16 < 0$$

$$\pi_{12} = \frac{\partial}{\partial Q_2}(\pi_1) = \frac{\partial}{\partial Q_2}(80 - 4Q_2 - 16Q_1) = -4 < 0$$

$$\pi_{21} = \frac{\partial}{\partial Q_1}(\pi_2) = \frac{\partial}{\partial Q_1}(50 - 4Q_1 - 10Q_2) = -4 < 0$$

$$\pi_{22} = \frac{\partial}{\partial Q_2}(\pi_2) = \frac{\partial}{\partial Q_2}(50 - 4Q_1 - 10Q_2) = -10 < 0$$

$\Rightarrow \ \pi_{11},\ \pi_{22} < 0$	$(\pi_{11})(\pi_{22}) > (\pi_{12})^2$
$-16,\ -10 < 0$	$(-16)(-10) > (-4)(-4)$
	$160 > 16$

Maximum exists.

– Maximized profits: π_{max} at $Q_1^* = 4.17$ & $Q_2^* = 3.33$

$$\pi_{max} = 80(4.17) + 50(3.33) - 4(4.17)(3.33) - 8(4.17)^2 - 5(3.33)^2$$

$$\boxed{\pi_{max} = 249.99}$$

Firms under monopolistic competition should produce 4.17 and 3.33 units of Q_1 and Q_2 to maximize the profits upto 249.99 units.

TOPIC 180: PROFIT MAXIMIZATION OF FIRM PRODUCING SUBSTITUTE GOODS

Consider the producer producing two substitutes.

$P_1 = 130 - 4Q_1 - Q_2$ & $P_2 = 160 - 2Q_1 - 5Q_2$

$C = 2Q_1^2 + 2Q_1Q_2 + 4Q_2^2$

Above-mentioned are the inverse demand functions and cost functions.

- Profit function can be developed using the given information.

$\pi = R_1 + R_2 - C$

$\pi = P_1Q_1 + P_2Q_2 - C$

$= (130 - 4Q_1 - Q_2)Q_1 + (160 - 2Q_1 - 5Q_2)Q_2 - (2Q_1^2 + 2Q_1Q_2 + 4Q_2^2)$

$= 130Q_1 - 4Q_1^2 - Q_1Q_2 + 160Q_2 - 2Q_1Q_2 - 5Q_2^2 - 2Q_1^2 - 2Q_1Q_2 - 4Q_2^2$

Simplifying we get

$\pi = 130Q_1 + 160Q_2 - 5Q_1Q_2 - 6Q_1^2 - 9Q_2^2$

First order-conditions

$$\frac{\partial(\pi)}{\partial Q_1} = \frac{\partial}{\partial Q_1}\left(130Q_1 + 160Q_2 - 5Q_1Q_2 - 6Q_1^2 - 9Q_2^2\right) = 0$$

$\pi_1 = 130 - 5Q_2 - 12Q_1 = 0$ —— (1)

$\pi_2 = 160 - 5Q_1 - 18Q_2 = 0$ —— (2)

eq ① and eq ② can be re-written as follows:

$$Q_2 = \frac{130 - 12Q_1}{5} \quad \& \quad Q_1 = \frac{160 - 18Q_2}{5}$$

$$Q_1 = \frac{1}{5}\left\{160 - 18\left(\frac{130 - 12Q_1}{5}\right)\right\} \quad \Big| \quad Q_1^* = 8.06, Q_2^* = 6.65$$

Second order conditions:

$$\pi_{11} = \frac{\partial(\pi_1)}{\partial Q_1} = \frac{\partial}{\partial Q_1}(130 - 5Q_2 - 12Q_1) = -12 < 0$$

$$\pi_{12} = \frac{\partial(\pi_1)}{\partial Q_2} = \frac{\partial}{\partial Q_2}(130 - 5Q_2 - 12Q_1) = -5 < 0$$

$$\pi_{21} = \frac{\partial}{\partial Q_1}(\pi_2) = \frac{\partial}{\partial Q_1}(160 - 5Q_1 - 18Q_2) = -5 < 0$$

$$\pi_{22} = \frac{\partial}{\partial Q_2}(\pi_2) = \frac{\partial}{\partial Q_2}(160 - 5Q_1 - 18Q_2) = -18 < 0$$

Maximization requires: $\pi_{11}, \pi_{22} < 0$
$\& \ (\pi_{11})(\pi_{22}) > (\pi_{12})^2$

Here $-12, -18 < 0$
$\& \ (-12)(-18) > (-5)(-5)$
Therefore profit is maximized.

The maximized value of profit can be found by plugging-in the critical values of Q_1 & Q_2

$$\left[\pi\right]_{(8.06,\, 6.65)} = 130(8.06) + 160(6.65) - 5(8.06)(6.65) - 6(8.06)^2 - 9(6.65)^2$$

$$\pi_{max}(8.06, 6.65) = 1056.02 \text{ units.}$$

While producing 8.06 units of Q_1 and 6.65 units of Q_2 (which are actually substitutes, firm can maximize its profit to 1056.02 units.

Lesson 37

PROFIT MAXIMIZATION ANALYSIS (CONTINUED 1)

TOPIC 181: MARGINAL AND AVERAGE REVENUE ANALYSIS

- Marginal Revenue Curve under imperfect competition is -vely sloped.
- If MR is the marginal revenue and AR is average revenue, they can be written as follows:

$$AR = f(Q)$$

$$R = Q.\ f(Q)$$

$$MR = \frac{d(R)}{dQ} = \frac{d}{dQ}\left[Q.\ f(Q)\right]$$

Applying product theorem.

$$= \frac{d}{dQ}(Q).\ f(Q) + \frac{d}{dQ}\{f(Q)\}.\ Q$$

$$MR = f(Q) + f'(Q).Q$$

$$\boxed{MR = f(Q) + Q.f'(Q)}$$

Slope of MR can be found as well.

$$\frac{d}{dQ}(MR) = \frac{d}{dQ}\{f(Q) + Q.\ f'(Q)\}$$

$$= \frac{d}{dQ}\{f(Q)\} + \frac{d}{dQ}\{Q.\ f'(Q)\}$$

$$= f'(Q) + \left\{\frac{d}{dQ}(Q).\ f'(Q) + \frac{d}{dQ}\{f'(Q)\}.\ Q\right\}$$

$$= f'(Q) + f'(Q) + f''(Q).Q$$

$$\frac{d}{dQ}(MR) = 2f'(Q) + Q.\ f''(Q)$$

Under Imperfect competition AR is downward sloping. So MR tends to be negative.

Unit (q)	TR/q AR or Price	(Pq) TR	($TR_n - TR_{n-1}$) MR
1	10	10	10
2	9	18	8
3	8	24	6
4	7	28	4
5	6	30	2
6	5	30	0
7	4	28	− 2
8	3	24	− 4
9	2	18	− 6
10	1	10	− 8

Numerically speaking, if

$$AR = f(Q) = 8000 - 23Q + 1.1Q^2 - 0.018Q^3$$

$$MR = \frac{d}{dQ}(R).$$

$$R = (AR)Q.$$

$$R = (8000 - 23Q + 1.1Q^2 - 0.018Q^3)Q$$

$$\frac{d}{dQ}(R) = 8000Q - 23Q^2 + 1.1Q^3 + 0.018Q^4$$

$$\frac{d}{dQ}(R) = MR = 8000 - 46Q + 3.3Q^2 - 0.072Q^3.$$

$$\text{slope of } MR = \frac{d}{dQ}(MR)$$

$$\Rightarrow -46 + 6.6Q - 0.216Q^2$$

$$\text{slope of } MR = -0.216Q^2 + 6.6Q - 46$$

Solving the quadratic eq.

$$-0.216Q^2 + 6.6Q - 46 = 0$$

$a = -0.216 < 0 \longrightarrow$ Inverted U-shaped parabola.

$b = 6.6$

$c = -46$

through quadratic formula.

$$Q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$Q^* = 10.76$ or $Q^* = 19.79$

These values help in graphing the slope of the MR function.

TOPIC 182: SHORT RUN PRODUCTION FUNCTION ANALYSIS

A Short-Run Production Function doesn't include all factors of production as input-variables.

i.e. $Q = F(\bar{K}, L)$ [As K is considered constant in the short-run]

$Q = F(L)$

$\frac{dQ}{dL} = MP_L = F_L > 0$ (+ve relationship)

$\frac{d^2Q}{dL^2}$ = slope of $MP_L = F_{LL} < 0$ (Increasing at a decreasing rate i.e. diminishing returns)

Assuming a numerical example.

$$Q = -0.1L^3 + 6L^2 + 12L$$

$Q = f(\bar{K}, L)$ hence a short-run production function

If average product function is $AP_L = \frac{Q}{L}$.

$$AP_L = \frac{-0.1L^3 + 6L^2 + 12L}{L}$$

$$AP_L = -0.1L^2 + 6L + 12$$

Optimizing the AP_L function

[F.o.C]

$$\frac{d(AP_L)}{dL} = -0.2L + 6 = 0$$

$$\Rightarrow L^* = 30 \quad \text{[Critical value of Labour]}$$

[S.o.C]

$$\frac{d}{dL}\left(\frac{d\,AP_L}{dL}\right) = -0.2 < 0 \Rightarrow \text{Maximum exists.}$$

$$(AP_L)_{max} = AP_L(30) = -0.1(30)^2 + 6(30) + 12$$

$$\boxed{AP_{max} = 102}$$

If marginal product function is $MP_L = \frac{dQ}{dL}$

$$MP_L = \frac{d}{dL}\left(-0.1L^3 + 6L^2 + 12L\right)$$

$$MP_L = -0.3L^2 + 12L + 12$$

Optimizing the MP_L function

$$\frac{d}{dL}(MP_L) = \frac{d}{dL}\left(-0.3L^2 + 12L + 12\right) = 0$$

$$= -0.6L + 12 = 0$$

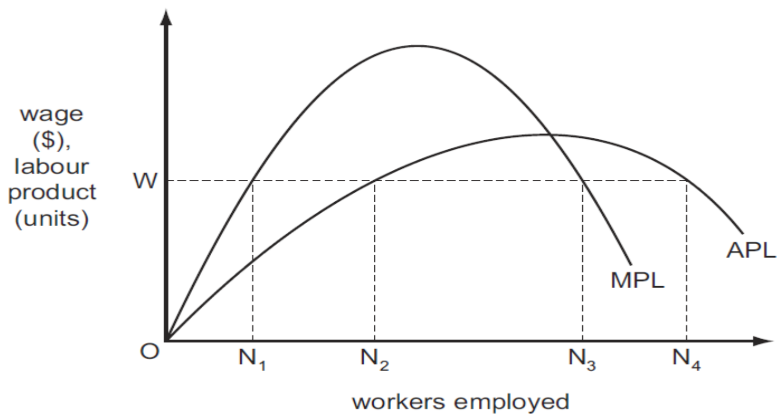
$$\Rightarrow L^* = 20 \quad \text{[Critical value of labour]}$$

$$\frac{d^2(MP_L)}{dL^2} = \frac{d}{dL}\left(\frac{d}{dL}(MP_L)\right) = \frac{d}{dL}\left(-0.6L + 12\right)$$

$$\frac{d^2}{dL^2}(MP_L) = -0.6 < 0 \Rightarrow \text{maximum exists.}$$

$$(MP_L)_{max} = MP_L(20) = -0.3(20)^2 + 12(20) + 12$$

$$\boxed{(MP_L)_{max} = 132}$$



TOPIC 183: TOTAL COST, TOTAL REVENUE AND PROFIT MAXIMIZATION

Assuming Cost and Demand functions of a firm's product Q.

$$C = \tfrac{1}{3}Q^3 - 7Q^2 + 111Q + 50$$

$$Q = 100 - P$$

In order to maximize the profit we need profit function.

$$\pi = R - C$$

$$= P \cdot Q - C$$

$Q = 100 - P$ Re-written $P = 100 - Q$ [Inverse demand function $P(Q)$]

$$\pi(Q) = (100 - Q)(Q) - (\tfrac{1}{3}Q^3 - 7Q^2 + 111Q + 50)$$

$$\pi(Q) = 100Q - Q^2 - \tfrac{1}{3}Q^3 + 7Q^2 - 111Q - 50$$

$$\pi(Q) = -\tfrac{1}{3}Q^3 + 6Q^2 - 11Q - 50$$

Applying First order condition.

$$\frac{d}{dQ}\pi(Q) = \frac{d}{dQ}\left(-\tfrac{1}{3}Q^3 + 6Q^2 - 11Q - 50\right) = 0$$

$$\pi'(Q) = -Q^2 + 12Q - 11 = 0$$

$$Q^2 - 12Q + 11 = 0$$

$$Q^2 - 11Q - Q + 11 = 0$$

$$Q(Q - 11) - (Q - 11) = 0$$

$$(Q - 1)(Q - 11) = 0$$

Either $Q = 1$, $Q = 11$

Applying second order condition.

$$\frac{d}{dQ}\{\pi'(Q)\} = -2Q + 12$$

$Q = 1$	$Q = 11$
$\pi''(Q) = -2(1) + 12$	$\pi'(Q) = -2(11) + 12$
$= 10 > 0$	$= -10 < 0$
minimum	maximum

Maximized profit π_{max} can be found using the critical value of Q i.e. $Q^* = 11$.

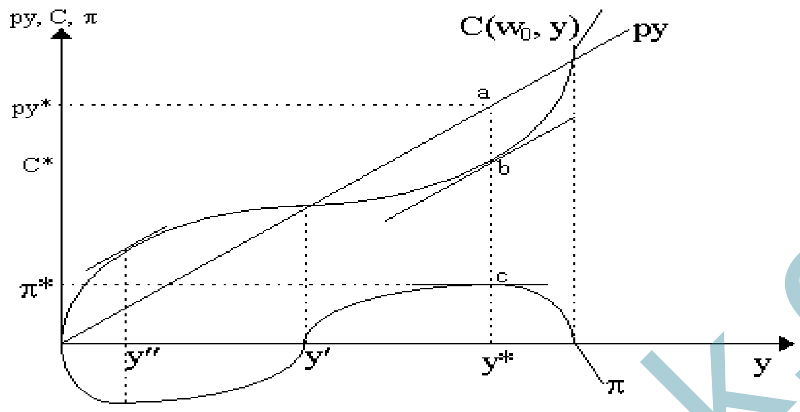
$$\pi(11) = -\tfrac{1}{3}(11)^3 + 6(11)^2 - 11(11) - 50$$

$$\pi_{max}(11) = 111.33 \text{ unit on producing 11 units}$$

Lesson 38

PROFIT MAXIMIZATION ANALYSIS (CONTINUED 2)

TOPIC 184: QUADRATIC PROFIT FUNCTION ANALYSIS



Consider following Quadratic Profit Function.

$$\pi(Q) = h(Q^2) + jQ + k$$

Profit at no output.

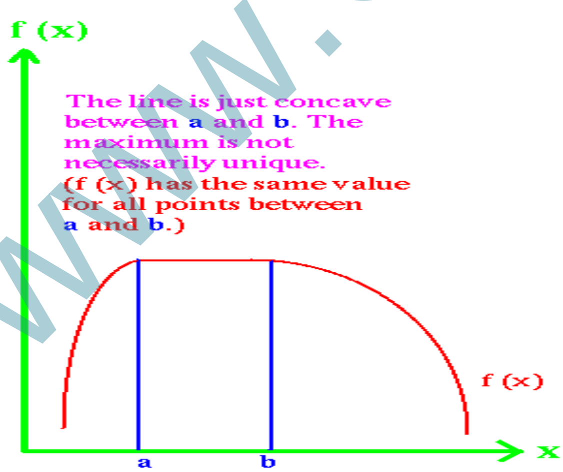
$\Rightarrow Q = 0$

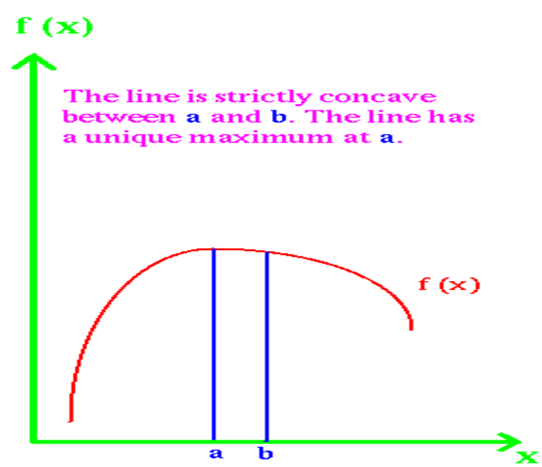
$$\pi(0) = h(0)^2 + j(0) + k$$

$$\pi(0) = k$$

For the $\pi(0)$ to be meaningful, $K<0$, implying that $\pi(0)<0$ — a loss.

K can also be looked at as fixed cost.





Concavity of the profit function

$\pi(Q) = hQ^2 + jQ + k$

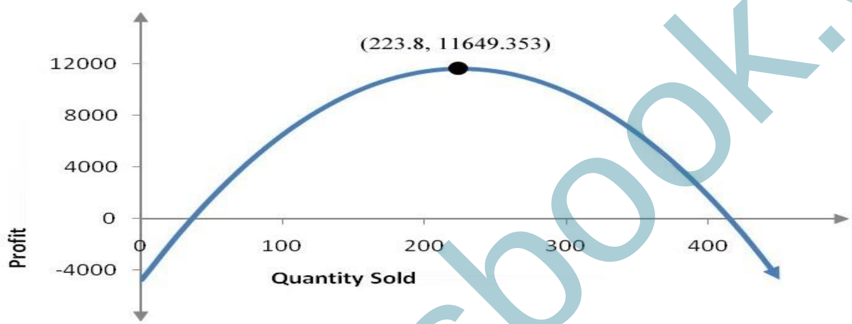
$\pi'(Q) = 2hQ + j$

$\pi''(Q) = 2h \gtrless 0$

For concavity $\frac{d^2}{dx^2}(y) = f''(x) < 0$

$\Rightarrow 2h < 0$

$h < 0$ parametric restriction for concavity of profit function.



Critical Value of output

$\pi'(Q) = 0 \Rightarrow 2hQ + j = 0 \Rightarrow Q^* = -j/2h$

$\pi''(Q) = 2h \gtrless 0$ since $h < 0$, $\pi''(Q) < 0$

Implying a maximum

& $Q^* > 0$ as $h < 0$

TOPIC 185: OPTIMIZATION OF EXPONENTIAL REVENUE FUNCTION

Considering an inverse demand function

$$P = 12.50\, e^{-0.005Q}$$

$$R = \left(12.50\, e^{-0.005Q}\right) Q.$$

$$R'(Q) = 12.50 \frac{d}{dQ}\left(e^{-0.005Q} \cdot Q\right)$$

Product rule & exponential rule

$$\frac{d}{dx}\left\{e^{f(x)}\right\} = f'(x) \cdot e^{f(x)}$$

$$R'(Q) = 12.50\left\{\frac{d}{dQ}\left(e^{-0.005Q}\right) \cdot Q + \frac{d}{dQ}(Q) \cdot e^{-0.005Q}\right\}$$

$$= 12.50\left\{(-0.005) \cdot e^{-0.005Q} \cdot Q + 1 \cdot e^{-0.005Q}\right\}$$

$$= 12.50\left\{e^{-0.005Q}(-0.005 \cdot Q + 1)\right\}$$

$$= 12.50\left\{e^{-0.005Q}(1 - 0.005Q)\right\} = 0$$

$$\Rightarrow 1 - 0.005Q = 0$$

$$Q^* = \frac{1}{0.005} = 200$$

$$P^* = 12.50\, e^{-0.005(200)}$$

$$P^* = 4.60$$

$$R''(Q) = 12.50 \frac{d}{dQ}\left\{e^{-0.005Q}(1 - 0.005Q)\right\}$$

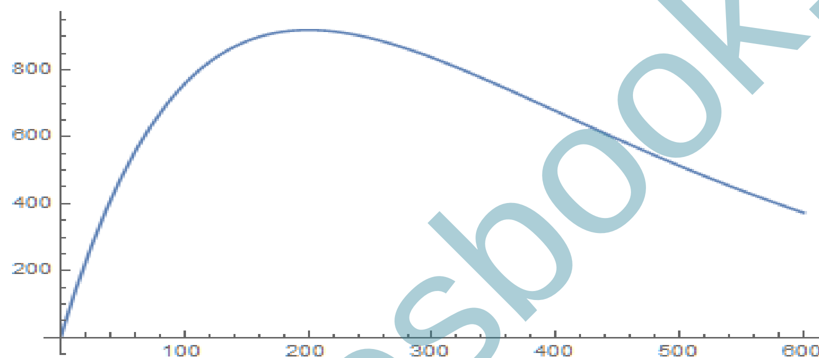
$$= 12.50\left\{(-0.005)\, e^{-0.005Q}(1 - 0.005Q) + (-0.005)\, e^{-0.005Q}\right\}$$

$$= 12.50(-0.005)\left\{e^{-0.005}\right\}(1 - 0.005Q + 1)$$

$$= (-0.005)\left(12.50\, e^{-0.005Q}\right)(2 - 0.005Q)$$

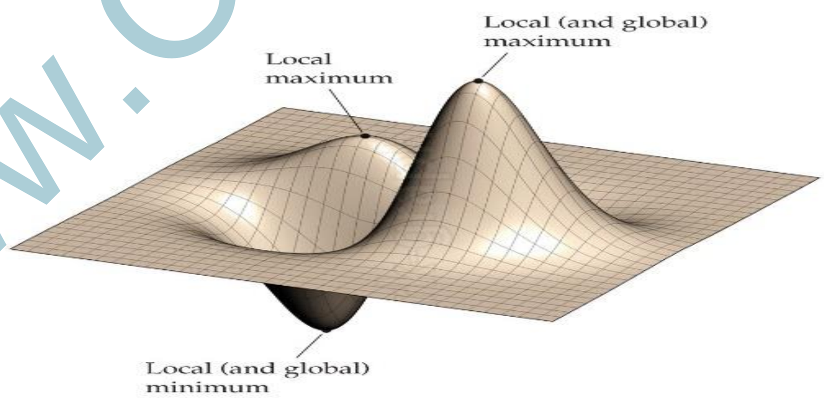
$$R''(Q) = (-0.005)\left(12.50\, e^{-0.005Q}\right)(2 - 0.005Q)$$

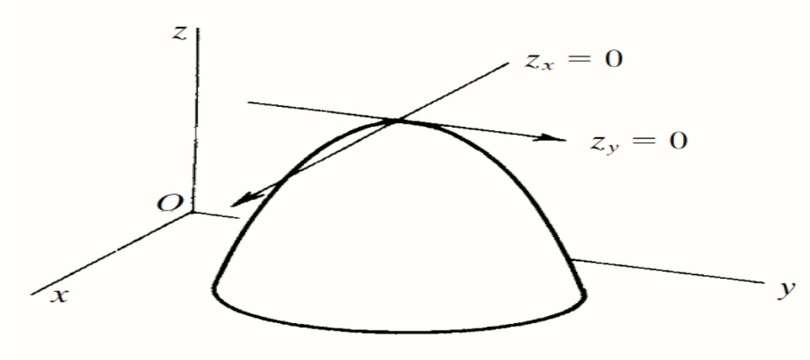
$$R''(200) = (-0.005)(12.50\, e^{-0.005(200)})(2 - 0.005(200))$$
$$= (-0.005)(12.50)\, e^{-1}(1)$$
$$= (-0.005)(12.50)(0.367879)$$
$$R''(200) = -0.02299 < 0 \quad (\text{maximum})$$
$$R(200) = 12.50\, e^{-0.005(200)} . (200)$$
$$= 12.50 . e^{-1} . 200$$
$$R(200) = 920 \quad \text{Maximized Revenue.}$$

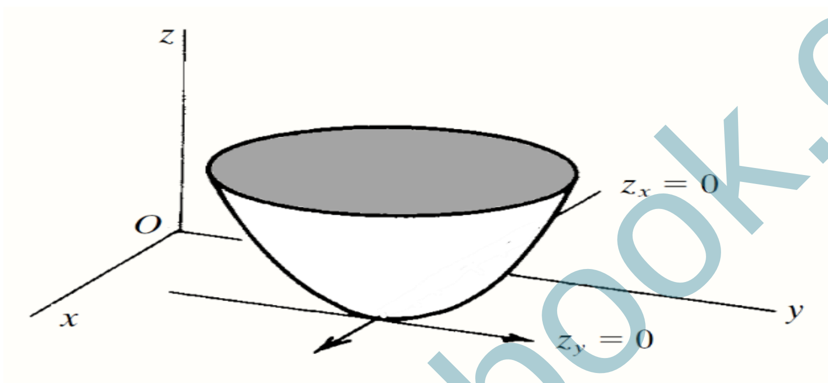


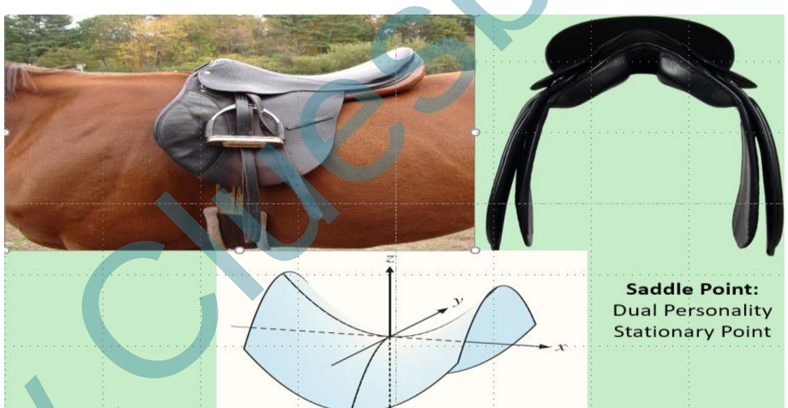
TOPIC 186: OPTIMIZATION OF MORE THAN ONE CHOICE VARIABLE

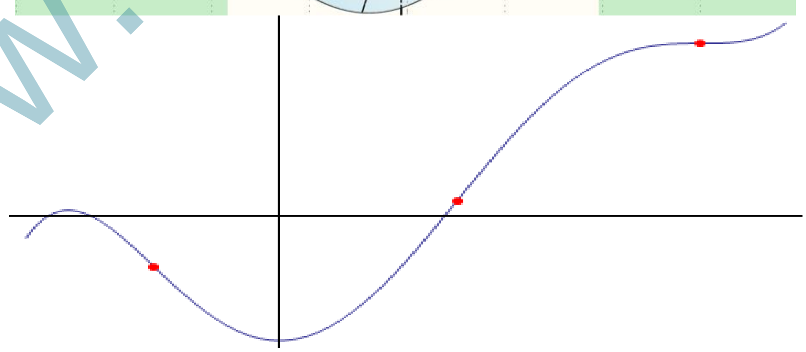
Two (or more) independent variables can exist.
For instance, $z = f(x, y)$
Surface (or hypersurface).
Peaks of domes & bottoms bowls can exist.




z
$z_x = 0$
$z_y = 0$
O
x
y


z
O
$z_x = 0$
x
y
$z_y = 0$


Saddle Point:
Dual Personality
Stationary Point



Conditions for Relative Extremum: $z = f(x, y)$		
Condition	**Maximum**	**Minimum**
1st Order (Necessary)	$f_x = f_y = 0$	$f_x = f_y = 0$
2nd Order(Necessary)	$f_{xx}, f_{yy} < 0$	$f_{xx}, f_{yy} > 0$
2nd Order (Sufficient)	$f_{xx} \cdot f_{yy} > (f_{xy})^2$	$f_{xx} \cdot f_{yy} > (f_{xy})^2$
Conditions for Inflection and Saddle Points:		
	Inflection Point	**Saddle Point**
2nd Order(Necessary)	$f_{xx}, f_{yy} < 0$ **OR** $f_{xx}, f_{yy} > 0$	$f_{xx} < 0, f_{yy} > 0$
2nd Order (Sufficient)	$f_{xx} \cdot f_{yy} < (f_{xy})^2$	$f_{xx} \cdot f_{yy} < (f_{xy})^2$
Conditions for Inconclusiveness of the Test:		
2nd Order (Sufficient)	$f_{xx} \cdot f_{yy} = (f_{xy})^2$	N.A.
Note: By Young's Theorem: $f_{xy} = f_{yx} \Rightarrow f_{xy} \cdot f_{yx} = (f_{xy})^2$		

- **Numerical Example**

$z(x, y) = 2y^3 - x^3 + 147x - 54y + 12$

1st order conditions:

$$z_x = -3x^2 - 147 = 0$$
$$\Rightarrow x = \pm 7$$
$$z_y = 6y^2 - 54 = 0$$

$\Rightarrow y = \pm 3$

Critical Points: $(\pm 7, \pm 3) \Rightarrow \{(7,\ 3), (7, -3), (-7,\ 3), (-7, -3)\}$

$$z_{xx} = -6x, \quad z_{yy} = 12y$$

A. $z_{xx}(7,\ 3) = -42,\ z_{yy}(7,\ 3) = 36$

B. $z_{xx}(7,\ -3) = -42,\ z_{yy}(7,\ -3) = -36$

C. $z_{xx}(-7,\ 3) = 42,\ z_{yy}(-7,\ 3) = 36$

D. $z_{xx}(-7,\ -3) = 42,\ z_{yy}(-7,\ -3) = -36$

POINT - I: $(x, y) = (7,\ 3)$

$$(z_{xx} = -42 < 0,\ z_{yy} = 36 > 0)$$

Different signs of z_{xx} & z_{yy} $\Rightarrow$ **Saddle** pt.

POINT - II: $(x, y) = (7,\ -3)$

$$(z_{xx} = -42 < 0,\ z_{yy} = -36 > 0)$$

$-ve$ z_{xx} & z_{yy} $\Rightarrow$ Maximum/inflection

$$z_{xx} \cdot z_{yy} \lesseqgtr (z_{xy})^2 \Rightarrow (-42)(-36) > 0^2$$

At $(7,\ -3)$ **Maximum** is confirmed.

POINT - III: $(x, y) = (-7,\ 3)$

$$\left(z_{xx} = 42 < 0,\ \ z_{yy} = 36 > 0\right)$$

$-ve\ z_{xx}$ & z_{yy}:Minimum/inflection

$$z_{xx} \cdot z_{yy} \lesseqgtr \left(z_{xy}\right)^2 \Rightarrow (42)(36) > 0^2$$

At $(7,\ -3)$ **Minimum** is confirmed.

POINT - IV: $(x, y) = (-7, -3)$

$$\left(z_{xx} = 42 < 0,\ \ z_{yy} = -36 > 0\right)$$

Different signs of z_{xx} & $z_{yy} \Rightarrow$ **Saddle** pt.

D.I.Y

1. $\boldsymbol{z = 48y - 3x^2 - 6xy - 2y^2 + 72x}$

2. $\boldsymbol{z = x^3 - \ 3xy^2}$

TOPIC 187: ECONOMIC APPLICATION ON MULTI-PRODUCT FIRM

Considering a firm in pure (perfect) competition product multi (two) products.

- Since the firm is in pure competition, the prices are considered to be exogenous.
  i.e. P_{10} and P_{20} for the two goods, respectively and 'o' in the subscript shows the autonomous value of prices
- Revenue for such a firm shall be:
  $$R_1 = P_{10} \cdot Q_1 + P_{20} Q_2$$
- Cost function:
  $$C = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2$$
  $$C = f(Q_1, Q_2)$$

– Profit function:

$$\pi = R_1 - C$$

$$\pi = P_{10} Q_1 + P_{20} Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2$$

F.O.C

$$\pi_1 = P_{10} - 4Q_1 - Q_2 = 0 \Rightarrow 4Q_1 + Q_2 = P_{10}$$

$$\pi_2 = P_{20} - Q_1 - 4Q_2 = 0 \Rightarrow Q_1 + 4Q_2 = P_{20}$$

$$\Rightarrow Q_1^* = \frac{4P_{10} - P_{20}}{15}, \quad Q_2^* = \frac{4P_{20} - P_{10}}{15}$$

Assuming $P_{10} = 12$ & $P_{20} = 18$

we get the critical values of Q_1^* and Q_2^*

$$Q_1^* = \frac{4(12) - 18}{15} = 2$$

$$Q_2^* = \frac{4(18) - 12}{15} = 4$$

S.O.C

$$\pi_{11} = \frac{\partial}{\partial Q_1}(\pi_1) = \frac{\partial}{\partial Q_1}(P_{10} - 4Q_1 - Q_2) = -4$$

$$\pi_{12} = \frac{\partial}{\partial Q_2}(\pi_1) = \frac{\partial}{\partial Q_2}(P_{10} - 4Q_1 - Q_2) = -1$$

$$\pi_{21} = \frac{\partial}{\partial Q_1}(\pi_2) = \frac{\partial}{\partial Q_1}(P_{20} - Q_1 - 4Q_2) = -1$$

$$\pi_{22} = \frac{\partial}{\partial Q_2}(\pi_2) = \frac{\partial}{\partial Q_2}(P_{20} - Q_1 - 4Q_2) = -4$$

π_{11} & $\pi_{22} < 0$ $[-4, -4]$

& $(\pi_{11})(\pi_{22}) > (\pi_{12})^2$ $[(-4)(-4) > (-1)^2]$

Hence profit of a multiproduct firm operating under pure/perfect competition is maximized while producing 2 and 4 units

– The maximized value of profit is 48.

TOPIC 188: ECONOMIC APPLICATION ON MULTI-PLANT FIRM

Consider a firm has two factories; one in China other in USA.

Output in China-factory is termed as x.

" " USA-factory " " " y.

Cost of product shall be

$C(x,y) = C_1(x) + C_2(y)$

$C(x) = \frac{x^2}{60} - 10x + 250$

$C(y) = \frac{y^2}{150} - 50y - 100$

$C(x,y) = \frac{x^2}{60} - 10x + \frac{y^2}{150} - 50y + 150$

F.o.C $C_x = \frac{\partial}{\partial x}(C(x,y)) = \frac{\partial}{\partial x}\left(\frac{x^2}{60} - 10x + \frac{y^2}{150} - 50y + 150\right)$

$C_x = \frac{x}{30} - 10 = 0$ —— (1)

$C_y = \frac{\partial}{\partial y}(C(x,y)) = \frac{\partial}{\partial y}\left(\frac{x^2}{60} - 10x + \frac{y^2}{150} - 50y + 150\right)$

$C_y = \frac{y^2}{50} - 50 = 0$ —— (2)

Solving eq (1) & eq (2) simultaneously.

$x^* = 300$, $y^* = 50$ Critical values of x & y.

S.o.C $C_{xx} = \frac{\partial}{\partial x}(C_x) = \frac{\partial}{\partial x}\left(\frac{x}{30} - 10\right) = 1/30 > 0$

$C_{xy} = \frac{\partial}{\partial y}(C_x) = \frac{\partial}{\partial y}\left(\frac{x}{30} - 10\right) = 0$

$C_{yy} = \frac{\partial}{\partial y}(C_y) = \frac{\partial}{\partial y}\left(\frac{y^2}{50} - 50\right) = y/25 > 0$

$C_{yx} = \frac{\partial}{\partial x}(C_y) = \frac{\partial}{\partial x}\left(y^2/50 - 50\right) = 0$

C_{xx} & $C_{yy} > 0$ $\left[\frac{1}{30}, \frac{y}{25}\right]$ where $y > 0$

$(C_{xx})(C_{yy}) > (C_{xy})^2$ $\left[\left(\frac{1}{30}\right)\left(\frac{y}{25}\right) > (0)^2\right]$

– So cost is minimized at $x = 300$ and $y = 50$

Lesson 39

PROFIT MAXIMIZATION ANALYSIS (CONTINUED 3)

TOPIC 189: PRICE DISCRIMINATION BY MONOPOLY

Consider a monopolist discriminating in 3-markets

i.e. $Q = Q_1 + Q_2 + Q_3$.

⇒ Three different demand structures.

Cost and revenue are determined by these different structures.

different demand prevailing in different mkt.

$$R(Q_1, Q_2, Q_3) = R(Q) = R_1(Q_1) + R_2(Q_2) + R_3(Q_3)$$

$$C(Q_1, Q_2, Q_3) = C(Q)$$ ← Same cost structure for all outputs.

$$\pi = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q)$$

$$\frac{\partial \pi}{\partial Q_1} = \pi_1 = R_1'(Q_1) + 0 + 0 - \frac{\partial C(Q)}{\partial Q}\left(\frac{\partial Q}{\partial Q_1}\right) = 0$$

$$\frac{\partial \pi}{\partial Q_1} = \pi_1 = R_1'(Q_1) - C'(Q) \cdot (1) = 0$$

$$\pi_1 = R_1'(Q_1) - C'(Q) = 0$$

$$\Rightarrow R_1'(Q_1) = C'(Q) \quad \text{—(a)}$$

Similarly for π_2 and π_3 $\left(\frac{\partial \pi}{\partial Q_2} \& \frac{\partial \pi}{\partial Q_3}\right)$

$$R_2'(Q_2) = C'(Q) \quad \text{—(b)}$$

&

$$R_3'(Q_3) = C'(Q) \quad \text{—(c)}$$

Combining eq. (a), (b) and (c)

$$R_1'(Q_1) = R_2'(Q_2) = R_3'(Q_3) = C'(Q)$$

OR

$$MR_1 = MR_2 = MR_3 = MC$$

Such levels of Q_1, Q_2, & Q_3 should be chosen that would achieve this condition to maximize the profits of the monopolist.

D.I.Y Numerical

TOPIC 190: PRICE DISCRIMINATION BY MONOPSONY

Consider a monopsonist with demand for two types of labour to produce (Q) output.

i.e. $Q = L_1 + L_2$ (a simple production function with $\bar{K}$)

The wage rates to attract a given labour supply.

$$w_1 = \alpha_1 + \beta_1 L_1 \quad \&$$

$$w_2 = \alpha_2 + \beta_2 L_2$$

The monopsonist is assumed to be competitive in its output market. $\Rightarrow$ P is fixed.

$$\pi = P.Q - (w_1 L_1 + w_2 L_2)$$

$$= P(L_1 + L_2) - \{(\alpha_1 + \beta_1 L_1)L_1 + (\alpha_2 + \beta_2 L_2)L_2\}$$

$$= PL_1 + PL_2 - \{\alpha_1 L_1 + \beta_1 L_1^2 + \alpha_2 L_2 + \beta_2 L_2^2\}$$

$$= PL_1 + PL_2 - \alpha_1 L_1 - \beta_1 L_1^2 - \alpha_2 L_2 - \beta_2 L_2^2$$

$$= PL_1 - \alpha_1 L_1 + PL_2 - \alpha_2 L_2 - \beta_1 L_1^2 - \beta_2 L_2^2$$

$$= (P - \alpha_1)L_1 + (P - \alpha_2)L_2 - \beta_1 L_1^2 - \beta_2 L_2^2$$

$$\pi = (P - \alpha_1)L_1 - \beta_1 L_1^2 + (P - \alpha_2)L_2 - \beta_2 L_2^2$$

$$\pi_1'(L_1, \bar{L}_2) = (P - \alpha_1) - 2\beta_1 L_1 = 0 \quad \text{--- (1)}$$

$$\pi_2'(\bar{L}_1, L_2) = (P - \alpha_2) - 2\beta_2 L_2 = 0 \quad \text{--- (2)}$$

Rearranging eq. ① & eq ②

$$L_1^* = \left(\frac{P-\alpha_1}{2\beta_1}\right), \quad L_2^* = \left(\frac{P-\alpha_2}{2\beta_2}\right)$$

& maximized profit $\pi_{max}(L_1^*, L_2^*)$

$$\pi^*_{max} = (P-\alpha_1)L_1^* - \beta_1 L_1^{*2} + (P-\alpha_2)L_2^* - \beta_2 L_2^{*2}$$

$$= (P-\alpha_1)\left(\frac{P-\alpha_1}{2\beta_1}\right) - \beta_1\left(\frac{P-\alpha_1}{2\beta_1}\right)^2 + (P-\alpha_2)\left(\frac{P-\alpha_2}{2\beta_2}\right) - \beta_2\left(\frac{P-\alpha_2}{2\beta_2}\right)^2$$

$$= \frac{(P-\alpha_1)^2}{2\beta_1} - \frac{\beta_1}{4\beta_1^2}(P-\alpha_1)^2 + \frac{(P-\alpha_2)^2}{2\beta_2} - \frac{\beta_2}{4\beta_2^2}(P-\alpha_2)^2$$

$$= \frac{(P-\alpha_1)^2}{2\beta_1} - \frac{(P-\alpha_1)^2}{4\beta_1} + \frac{(P-\alpha_2)^2}{2\beta_2} - \frac{(P-\alpha_2)^2}{4\beta_2}$$

$$= \frac{2(P-\alpha_1)^2 - (P-\alpha_1)^2}{4\beta_1} + \frac{2(P-\alpha_2)^2 - (P-\alpha_2)^2}{4\beta_2}$$

$$\pi^*_{max} = \frac{(P-\alpha_1)^2}{4\beta_1} + \frac{(P-\alpha_2)^2}{4\beta_2}$$

Corresponding wages:

$$w_1^* = \alpha_1 + \beta_1 L_1^*$$

$$= \alpha_1 + \beta_1\left(\frac{P-\alpha_1}{2\beta_1}\right)$$

$$= \alpha_1 + \frac{(P-\alpha_1)}{2}$$

$$= \frac{2\alpha_1 + P - \alpha_1}{2}$$

$$= \frac{\alpha_1 + P}{2}$$

$$w_1^* = \frac{1}{2}(P+\alpha_1)$$

D.I.Y for w_2^*

however, virtue of symmetry implies.

$$w_2^* = \frac{1}{2}(P+\alpha_2)$$

TOPIC 191: INPUT DECISION OF A FIRM

For an input to be employed.

$$MR_i = MC_i \quad \begin{cases} MR_K = MC_K \\ MR_L = MC_L \end{cases}$$

For Capital.

$$R = PQ$$
$$= (100 - Q)Q$$
$$= 100Q - Q^2$$
$$= 100K^{0.5}L^{0.5} - (K^{0.5}L^{0.5})^2$$
$$R = 100K^{0.5}L^{0.5} - KL$$
$$MR_K = \frac{\partial}{\partial K}[100K^{0.5}L^{0.5} - KL]$$
$$= 100(0.5)K^{-0.5}L^{0.5} - L$$
$$MR_K = 50\frac{\sqrt{L}}{\sqrt{K}} - L$$
$$\& \quad MC_K = \frac{\partial(C)}{\partial K} = \frac{\partial}{\partial K}[L + 4K]$$
$$MC_K = 4$$
$$\Rightarrow MR_K = MC_K \Rightarrow 50\sqrt{\frac{L}{K}} - L = 4 \quad \text{——— (1)}$$

Similarly for labour $MR_L = MC_L$

$$MR_L = \frac{\partial}{\partial L}[R] = \frac{\partial}{\partial L}[100K^{0.5}L^{0.5} - KL]$$
$$MR_L = 50\frac{\sqrt{K}}{\sqrt{L}} - K$$
$$MC_L = \frac{\partial}{\partial L}[C] = \frac{\partial}{\partial L}[L + 4K]$$
$$MC_L = 1$$
$$MR_L = MC_L$$
$$50\frac{\sqrt{K}}{\sqrt{L}} - K = 1 \quad \text{——— (2)}$$

Manipulating eq. ① & ② for simple solution

eq ①/L

$$\frac{50\sqrt{L}}{\sqrt{K}\,L} - \frac{L}{L} = \frac{4}{L}$$

$$\frac{50}{\sqrt{KL}} - 1 = \frac{4}{L} \quad \text{— ③}$$

eq. 2/K

$$\frac{50\sqrt{K}}{\sqrt{L}\cdot K} - \frac{K}{K} = \frac{1}{K}$$

$$\frac{50}{\sqrt{KL}} - 1 = \frac{1}{K} \quad \leftarrow ④$$

Subtracting eq ④ from eq ③

$$\frac{50}{\sqrt{KL}} - 1 = \frac{4}{L}$$

$$\mp\frac{50}{\sqrt{KL}} \mp 1 = \mp\frac{1}{K}$$

$$0 = \frac{4}{L} - \frac{1}{K}$$

$$\frac{4}{L} = \frac{1}{K}$$

$$\boxed{L = 4K}$$

Putting $L = 4K$ in eq ①

$$50\frac{\sqrt{L}}{\sqrt{K}} - L = 4$$

$$50\frac{\sqrt{4K}}{\sqrt{K}} - 4K = 4$$

$$50(2) - 4K = 4$$

$$2(25\times 2) - 4K = 4$$

$$4(25) - 4K = 4$$

$$4(25 - K) = 4$$

$$\boxed{K^* = 25}$$

Therefore

$$L = 4K$$

$$L = 4(25)$$

$$\boxed{L^* = 100}$$

Input decided by the firm.

TOPIC 192: PROFIT MAXIMIZATION OF TWO-PRODUCT FIRM

Given that:

$Q_1 = 40 - 2P_1 - P_2,\ Q_2 = 35 - P_1 - P_2,\ C = Q_1^2 + 2Q_2^2 + 10,\ P_1 = 5 + Q_2 - Q_1,\ P_2 = Q_1 - 2Q_2 + 30,$

$$R_1 = (5 + Q_2 - Q_1)Q_1$$

$$R_2 = (Q_1 - 2Q_2 + 30)Q_2$$

$$C = {Q_1}^2 + 2{Q_2}^2 + 10$$

$$\pi = (5 + Q_2 - Q_1)Q_1 + (Q_1 - 2Q_2 + 30)Q_2 - {Q_1}^2 - 2{Q_2}^2 - 10$$

$$\pi = -2{Q_1}^2 + 2Q_1Q_2 + 5Q_1 - 4{Q_2}^2 + 30Q_2 - 10$$

$$\pi_1 = -4Q_1 + 2Q_2 + 5 = 0$$
$$\pi_2 = 2Q_1 - 8Q_2 + 30 = 0$$
$$-4Q_1 + 2Q_2 + 5 = 0$$
$$2Q_1 - 8Q_2 + 30 = 0$$
$$\left(Q_1 = \frac{25}{7}, Q_2 = \frac{65}{14}\right)$$

$\boldsymbol{P_1 = 5 + \frac{65}{7} - \frac{25}{7} = \frac{75}{7}}$

$\boldsymbol{P_2 = \frac{25}{7} - 2\left(\frac{65}{14}\right) + 30 = \frac{170}{7}\ \pi = (-2)\left(\frac{25}{7}\right)^2 + 2\left(\frac{25}{7}\right)\left(\frac{65}{14}\right) + 5\left(\frac{25}{7}\right) - 4\left(\frac{65}{14}\right)^2 + 30\left(\frac{65}{14}\right) - 10 = \frac{480}{7}}$

Lesson 40

CONSTRAINED OPTIMIZATION ANALYSIS

TOPIC 193: COMPARATIVE-STATIC ASPECTS OF OPTIMIZATION

Example # 1 Reduced form Solutions

$Q_1^* = \frac{4P_{10}-P_{20}}{15}$

$Q_2^* = \frac{4P_{20}-P_{10}}{15}$

$P_{10}\ \&\ P_{20} \Rightarrow Q_1^*$

$P_{10}\ \&\ P_{20} \Rightarrow Q_2^*$

Need for comparative static analysis.

Partial differentiation of Q_1^* w.r.t. P_{10} & P_{20}.

$$\frac{\partial}{\partial P_{10}}(Q_1^*) = \frac{\partial}{\partial P_{10}}\left(\frac{4P_{10}-P_{20}}{15}\right) = \frac{4}{15}$$

$$\frac{\partial}{\partial P_{20}}(Q_1^*) = \frac{\partial}{\partial P_{20}}\left(\frac{4P_{10}-P_{20}}{15}\right) = \frac{-1}{15}$$

$$\frac{\partial}{\partial P_{10}}(Q_2^*) = \frac{\partial}{\partial P_{10}}\left(\frac{4P_{20}-P_{10}}{15}\right) = \frac{-1}{15}$$

$$\frac{\partial}{\partial P_{20}}(Q_2^*) = \frac{\partial}{\partial P_{20}}\left(\frac{4P_{20}-P_{10}}{15}\right) = \frac{4}{15}$$

$$\frac{\partial Q_1^*}{\partial P_{10}} = \frac{4}{15} \Rightarrow 1 \text{ unit} \uparrow \mathrm{P}_{10} \Rightarrow \frac{4}{15} \text{ unit} \uparrow Q_1^*$$

$$\frac{\partial Q_1^*}{\partial P_{20}} = \frac{-1}{15} \neq 0 \Rightarrow 1 \text{ unit} \uparrow \mathrm{P}_{20} \Rightarrow \frac{1}{15} \text{ unit} \downarrow Q_1^*$$

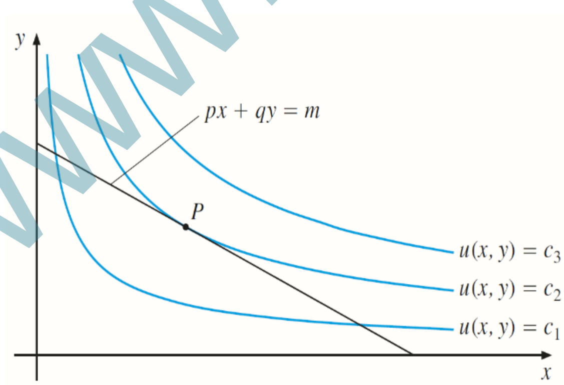
$$\frac{\partial Q_2^*}{\partial P_{10}} = \frac{-1}{15} \neq 0 \Rightarrow 1 \text{ unit} \uparrow \mathrm{P}_{10} \Rightarrow \frac{1}{15} \text{ unit} \downarrow Q_2^*$$

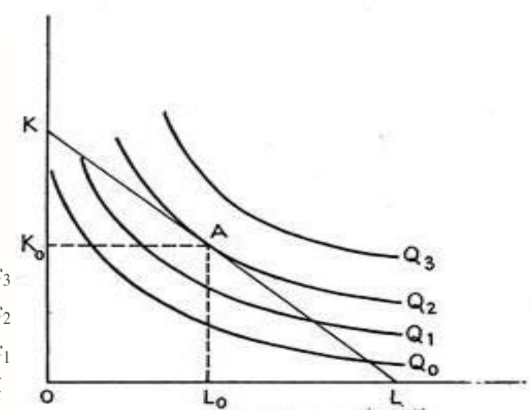
$$\frac{\partial Q_2^*}{\partial P_{20}} = \frac{4}{15} \Rightarrow 1 \text{ unit} \uparrow \mathrm{P}_{20} \Rightarrow \frac{4}{15} \text{ unit} \uparrow Q_2^*$$

- Technical relation in production

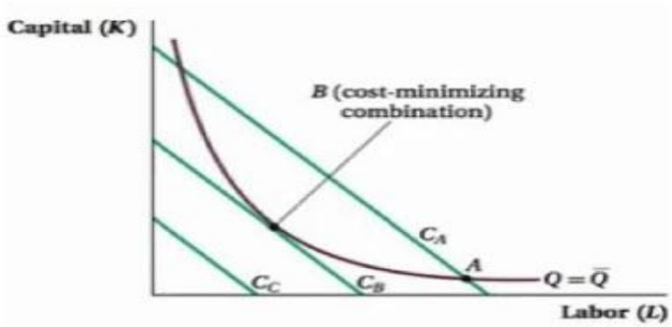
TOPIC 194: RATIONALE FOR CONSTRAINED OPTIMIZATION

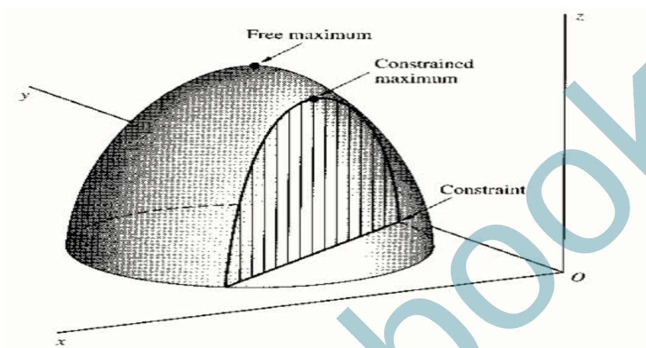
Constraints can be considered as old as scarcity.
a.k.a. Restraint, side relation, or subsidiary condition.
Narrows the domain (x) and hence range (y) of objective function (say $y = f(x)$).





Cost Minimization





Free optimum is higher than constrained optimum.
Sometimes both can be the same.
However, constrained optimum can't be higher than free optimum.

TOPIC 195: FINDING STATIONARY VALUES USING SUBSTITUTION/ELIMINATION METHOD

Assume an objective function:
$U(x,y) = xy + 2x$
A constraint function is:
$4x + 4y = 60$
Use constraint:
$y = 30 - 2x$
Substitute in objective function.
$U(x) = x(30 - 2x) + 2x$

$$U(x) = 32x - 2x^2$$

First order condition

$$U'(x) = 32 - 4x = 0$$
$$x^* = 8$$

$y^* = 30 - 2(8) = 14$
Critical values: $x^* = 8$ and $y^* = 14$
Substitute $x^*$and y^*in objective function:

$$U_{max}(x^*, y^*) = x^*y^* + 2x^*$$
$$U_{max}(8,14) = (8)(14) + 2(8)$$
$$U_{max}(8,14) = 128$$

TOPIC 196: FINDING STATIONARY VALUES USING METHOD OF LAGRANGE MULTIPLIER

Assume an objective function:

$$U(x, y) = xy + 2x$$

A constraint function is:

$$4x + 4y = 60$$

Forming Lagrange function

$$L(x, y, \lambda) = xy + 2x + \lambda(60 - 4x - 4y)$$

First of conditions:

$$L_x = \frac{\partial\{L(x,y,\lambda)\}}{\partial x} = y + 2 - 4\lambda = 0$$

$$L_y = \frac{\partial\{L(x,y,\lambda)\}}{\partial y} = x - 2\lambda = 0$$

$$L_\lambda = \frac{\partial\{L(x,y,\lambda)\}}{\partial \lambda} = 60 - 4x - 2y = 0$$

Rewriting:

$$y + 2 - 4\lambda = 0$$
$$x - 2\lambda = 0$$
$$60 - 4x - 2y = 0$$

Solving first two equation for λ.

$\lambda = \frac{y+2}{4}$ & $\lambda = \frac{x}{2}$ equation we get:

$$\frac{y+2}{4} = \frac{x}{2} \Rightarrow y = 2x - 2$$

Substituting in 3rd F.o.C.

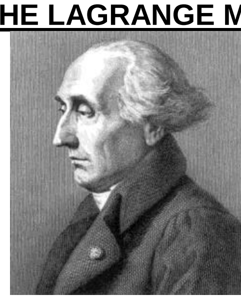
$$60 - 4x - 2(2x - 2) = 0$$
$$64 - 8x = 0 \Rightarrow x^* = 8$$
$$\Rightarrow y^* = 2x^* - 2$$
$$y^* = 14$$

Critical values: $x^* = 8$ and $y^* = 14$

Substitute $x^*$and y^*in objective function:

$$U_{max}(x^*, y^*) = x^*y^* + 2x^*$$
$$U_{max}(8, 14) = (8)(14) + 2(8)$$
$$U_{max}(8, 14) = 128$$

TOPIC 197: INTERPRETATION OF THE LAGRANGE MULTIPLIER



Introduced by Joseph-Louis Lagrange.

What if we want to predict the impact of per unit change in constraint on objective function.

Answer: Value of Lagrange multiplier.

a.k.a Shadow prices.

Considering example:

z = 4x^2+ 3xy + 6y^2 Subject to: $x + y = 56$

$\Longrightarrow x^* = 36, y^* = 20$ & $\lambda^* = 348$

1-unit increase (decrease) in the constant of the constraint (56) would cause z to increase (decrease) by approximately 348 units.

$z^*]_{x+y=56} = 4x^{*2} + 3x^*y^* + 6y^{*2} = 9744$

$z^*]_{x+y=56} = 4x^{*2} + 3x^*y^* + 6y^{*2}$

$$=4(36)^2 + 3(36)(20) + 6(20)^2$$

$$z^*]_{x+y=56} = 9744$$

Increasing the constant of constraint by 1 unit.

$z = 4x^2 + 3xy + 6y^2$ Subject to: $x + y = 57$

$\Rightarrow x^* = 36.64, y^* = 20.36 \;\&\; \lambda^* = 354.2$

$$z^*]_{x+y=57} = 4x^{*2} + 3x^*y^* + 6y^{*2}$$

$=4(36.64)^2 + 3(36.64)(20.36) + 6(20.36)^2$

$z^*]_{x+y=57} = 10095$

$(z^*]_{x+y=57} - z^*]_{x+y=56}) = 10095 - 9744 = 351$ **close to 348 as predicted by** $\lambda]_{x+y=56}$.

Economics Instances:

In utility maximization subject to a budget constraint: $U(x, y)$ sb. to $P_x x + P_y y = B$, λ will estimate the marginal utility of an extra PKR of income.

In output maximization subject to a cost constraint: $Q(K, L)$ sb. to $P_K K + P_L L = C$, λ will estimate the marginal product of an extra PKR of cost.

TOPIC 198: SECOND ORDER CONDITION: THE BORDERED HESSIAN

Attributed to German mathematician, *Ludwig Otto Hesse.*

In addition to algebra, determinant of matrices can also be used for 2nd order condition in constrained optimization.

$f(x, y)$ sb. to $g(x, y)$, is tested with 2nd order condition:

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix}$$

The standard form.

Order of a bordered principal minor equals the order of the principal minor being bordered (here, 2×2).

$|\bar{H}|$ represents a second bordered principal minor $|\bar{H}_2| = \begin{vmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{vmatrix}$

In algebraic form:

$$= (0) \cdot \begin{vmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{vmatrix} - (g_x) \cdot \begin{vmatrix} g_x & Z_{xy} \\ g_y & Z_{yy} \end{vmatrix} + (g_y) \cdot \begin{vmatrix} g_x & Z_{xx} \\ g_y & Z_{yx} \end{vmatrix}$$

$= -(g_x)(g_x \cdot Z_{yy} - Z_{xy} \cdot g_y) + (g_y) \cdot (g_x \cdot Z_{yx} - Z_{xx} \cdot g_y)$
$= -(g_x)^2 \cdot Z_{yy} - \mathbf{Z_{xy}} \cdot g_x \cdot g_y + g_x \cdot g_y \cdot \mathbf{Z_{yx}} - Z_{xx} \cdot (g_y)^2$
$= -(g_x)^2 \cdot Z_{yy} - \mathbf{Z_{xy}} \cdot g_x \cdot g_y + \mathbf{Z_{xy}} \cdot g_x \cdot g_y - Z_{xx} \cdot (g_y)^2$
$= -(g_x)^2 \cdot Z_{yy} - 2\,\mathbf{Z_{xy}} \cdot g_x \cdot g_y - Z_{xx} \cdot (g_y)^2$
For a function with n variables $f(x_1, x_2, \ldots, x_n)$, subject to $g(x_1, x_2, \ldots, x_n)$:

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & F_{11} & F_{12} & \cdots & F_{1n} \\ g_2 & F_{21} & F_{22} & \cdots & F_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & F_{n1} & F_{n2} & \cdots & F_{nn} \end{vmatrix}$$

Where, $|\bar{H}| = |\bar{H}_n|$, because of the $n \times n$ principal minor being bordered.
For maximum: $|\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; \ldots; (-1)^n|\bar{H}_n| > 0$
For maximum: $|\bar{H}_2| > 0; |\bar{H}_3| < 0; |\bar{H}_4| > 0; \ldots; (-1)^n|\bar{H}_n| > 0$
$(-1)^n$: Explained.
For even value of $n(= 4)$, final bordered principal minor:
$(-1)^4|\bar{H}_n| > 0 = |\bar{H}_n| > 0$
For odd value of $n(= 3)$, final bordered principal minor:
$(-1)^3|\bar{H}_n| > 0 = -|\bar{H}_n| < 0$
$(-1)^n$ helps to retain alternative signs.

$$z = 4x^2 + 3xy + 6y^2$$
$$x + y = 56$$
$$Z = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$$
$$Z_x = 8x + 3y - \lambda = 0$$
$$Z_y = 3x + 12y - \lambda = 0$$
$$Z_\lambda = 56 - x - y = 0$$

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 12 \end{vmatrix} = -14$$

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 12 \end{vmatrix}$$

$$|\bar{H}_2| = \begin{vmatrix} 0 & 1 \\ 1 & 8 \end{vmatrix} = -1, |\bar{H}| = |\bar{H}_3| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 12 \end{vmatrix} = -14$$

$$|\bar{H}_2| = -1 < 0, |\bar{H}| = |\bar{H}_3| = -14 < 0$$

Since, same signs of bordered principal minors occur, a minimum is evident.
Economic Instance: Two period model of utility.

Lesson 41

UTILITY MAXIMIZATION ANALYSIS

TOPIC 199: TWO PERIOD MODEL OF UTILITY

Considering a two period utility model.

$$U(x_1, x_2) = x_1 x_2$$

x_1 = consumption in period 1

x_2 = " " " 2.

Consumer is endowed with a budget 'B' in the beginning of period 1.

Since time is involved we choose a discount rate 'r.'

Discounting the consumption in period 2.

$$= \frac{x_2}{(1+r)^1} = \frac{x_2}{(1+r)}$$

- Forming the budget constraint from inter-temporal point of view.

$$B = x_1 + \frac{x_2}{1+r}$$

Optimizing by Lagrangian method.

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda\left(B - x_1 - \frac{x_2}{1+r}\right)$$

$$\mathcal{L}_1 = \frac{\partial}{\partial x_1}\left(x_1 x_2 + \lambda B - \lambda x_1 - \frac{\lambda x_2}{1+r}\right) = 0$$

$$\mathcal{L}_1 = x_2 - \lambda = 0 \quad \text{---} \ (1)$$

$$\mathcal{L}_2 = \frac{\partial}{\partial x_2}\left[x_1 x_2 + \lambda B - \lambda x_1 - \frac{\lambda x_2}{1+r}\right] = 0$$

$$\mathcal{L}_2 = x_1 - \frac{\lambda}{1+r} = 0 \quad \text{---} \ (2)$$

$$\mathcal{L}_\lambda = \frac{\partial}{\partial \lambda}\left[x_1 x_2 + \lambda B - \lambda x_1 - \frac{\lambda x_2}{1+r}\right] = 0$$

$$\mathcal{L}_\lambda = B - x_1 - \frac{x_2}{1+r} = 0 \quad \text{——— (3)}$$

Solving eq (1) & eq (2)

$$\lambda = x_2 \quad \& \quad \lambda = x_1(1+r)$$

$$x_2 = x_1(1+r) \quad \text{substituting in eq (3).}$$

$$B - x_1 - \frac{x_1(1+r)}{(1+r)} = 0$$

$$B - 2x_1 = 0$$

$$\boxed{x_1^* = \frac{B}{2}}$$

$$x_2^* = x_1^*(1+r)$$

$$\boxed{x_2^* = \frac{B}{2}(1+r)}$$

S.o.C

$$|\bar{H}| = \begin{vmatrix} 0 & B_1 & B_2 \\ B_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ B_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix}$$

$$B_1 = \frac{\partial}{\partial x_1}(B) = \frac{\partial}{\partial x_1}\left(x_1 + \frac{x_2}{1+r}\right) = 1$$

$$B_2 = \frac{\partial}{\partial x_2}(B) = \frac{\partial}{\partial x_2}\left(x_1 + \frac{x_2}{1+r}\right) = \frac{1}{1+r}$$

$$\mathcal{L}_{11} = \frac{\partial}{\partial x_1}(\mathcal{L}_1) = \frac{\partial}{\partial x_1}(x_2 - \lambda) = 0$$

$$\mathcal{L}_{12} = \frac{\partial}{\partial x_2}(\mathcal{L}_1) = \frac{\partial}{\partial x_2}(x_2 - \lambda) = 1$$

$$\mathcal{L}_{21} = \frac{\partial}{\partial x_1}(\mathcal{L}_2) = \frac{\partial}{\partial x_1}\left(x_1 - \frac{\lambda}{1+r}\right) = 1$$

$$\mathcal{L}_{22} = \frac{\partial}{\partial x_2}(\mathcal{L}_2) = \frac{\partial}{\partial x_2}\left(x_1 - \frac{\lambda}{1+r}\right) = 0$$

Plugging in the values in $|\bar{H}|$.

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & \frac{1}{1+r} \\ 1 & 0 & 1 \\ \frac{1}{1+r} & 1 & 0 \end{vmatrix}$$

$$|\bar{H}| = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ \frac{1}{1+r} & 0 \end{vmatrix} + \frac{1}{1+r}\begin{vmatrix} 1 & 0 \\ \frac{1}{1+r} & 1 \end{vmatrix}$$

$$= 0 - 1\left(-\frac{1}{1+r}\right) + \frac{1}{1+r}(1)$$

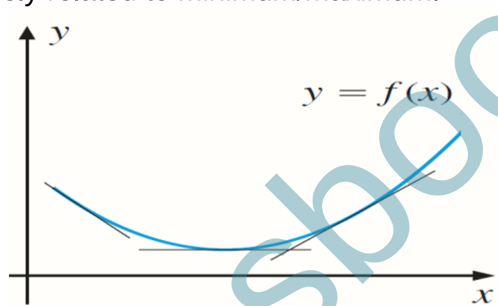
$$= \frac{1}{1+r} + \frac{1}{1+r}$$

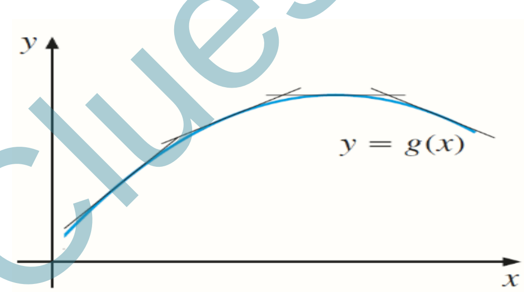
$$= \frac{1+1}{1+r}$$

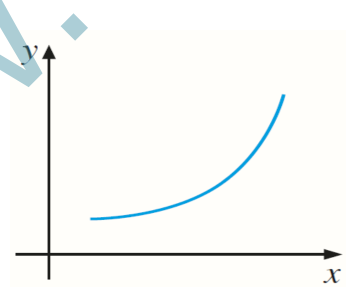
$$|\bar{H}| = \frac{2}{1+r} > 0 \quad \text{since } r > 0 \text{ the discount rate.}$$

TOPIC 200: CONVEXITY AND CONCAVITY USING SECOND ORDER DERIVATIVE

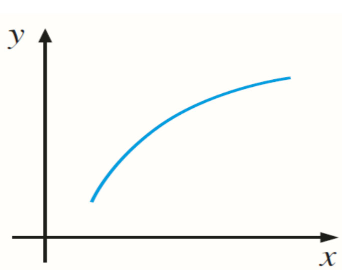
Convexity/concavity are closely related to minimum/maximum.



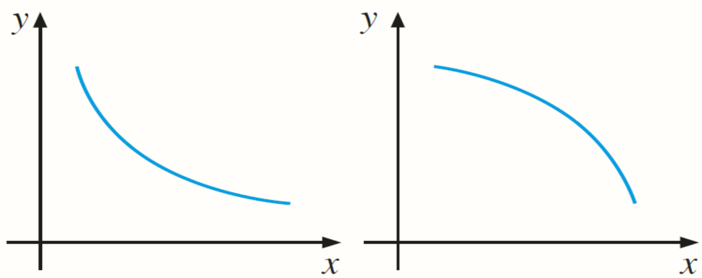




Increasing convex



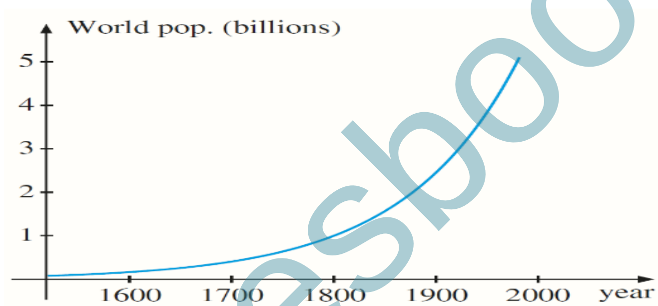
Increasing concave



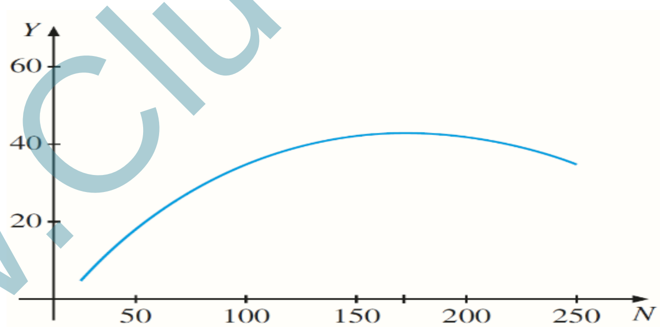
Decreasing convex Decreasing concave

f is continuous in the interval I and twice differentiable:
f is convex on $I \Leftrightarrow f''(x) > 0$ for all x in I
f is concave on $I \Leftrightarrow f''(x) < 0$ for all x in I

Numerical: $f(x) = x^2 - 2x + 2$
$f'(x) = 2x - 2;$
$f''(x) = 2 > 0 \Rightarrow \boldsymbol{Convex}$



World population

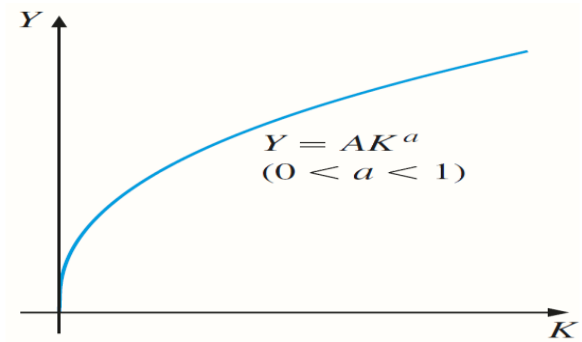


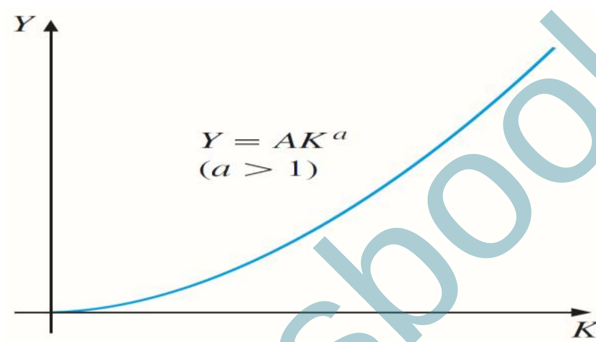
Wheat production

Single input production function:
$\boldsymbol{Y(K) = 100K^a}$
$$\boldsymbol{a > 0 = \begin{cases} 0 < a < 1, & Y(K) = 100K^{0.5} \\ a < 1, & Y(K) = 100K^{1.5} \end{cases}}$$

Case – I: $Y(K) = 100K^{\mathbf{0.5}}$
$Y'(K) = 50K^{-0.5}, Y''(K) = -25K^{-1.5} \Rightarrow Y''(1) = -25$
(Concave production function)

Case – II: $Y = 100K^{\mathbf{1.5}}$
$Y'(K) = 150K^{0.5}, Y''(K) = 75K^{-0.5} \Rightarrow Y''(1) = 75$
(Convex production function)





TOPIC 201: UTILITY MAXIMIZATION AND CONSUMER DEMAND: FIRST ORDER CONDITION

Hypothetical consumer with 2 goods $(\boldsymbol{x}, \boldsymbol{y})$, with continuous, +ve marginal utility $(\boldsymbol{U_x}, \boldsymbol{U_y} > \mathbf{0})$.
Market-determined prices $(\boldsymbol{P_x}, \boldsymbol{P_y})$ - exogenous
Budget constraint:

$$\boldsymbol{x \cdot P_x + y \cdot P_y = B}$$

Utility (objective) function:

$$\boldsymbol{U = U(x, y)}$$

Lagrangian function:
$\boldsymbol{U = U(x, y) + \lambda(B - xP_x + yP_y)}$
First order conditions:
$\boldsymbol{Z_x = U_x - \lambda P_x = 0 \Rightarrow \lambda = \frac{U_x}{P_x}}$
$\boldsymbol{Z_y = U_y - \lambda P_y = 0 \Rightarrow \lambda = \frac{U_y}{P_y}}$
$\boldsymbol{Z_\lambda = B - xP_x + yP_y = 0}$

Extracting value of $\boldsymbol{\lambda}$ from 1st & 2nd F.o.C and equating: $\boldsymbol{\frac{U_x}{P_x} = \frac{U_y}{P_y} = \lambda}$

$$\boldsymbol{\frac{U_x}{P_x} = \frac{U_y}{P_y} = \lambda^*}$$

Interpretation:
$\boldsymbol{\frac{U_x}{P_x} = \frac{U_y}{P_y}}$ is actually law of equi-marginal utility.

Marginal utility of money spend on each goods is equal.
λ^* can be termed as the marginal utility of (budget) money when utility is maximized.

$\frac{U_x}{P_x} = \frac{U_y}{P_y} \Rightarrow \frac{U_x}{U_y} = \frac{P_x}{P_y}$

Alternatively, $\left(\frac{U_x}{U_y}\right)$ & $\left(\frac{P_x}{P_y}\right)$:

$$\left(\frac{U_x}{U_y}\right) = MRTS_{(L,K)} = slope\ of\ IC.$$

$d\{U(x,y)\} = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy$

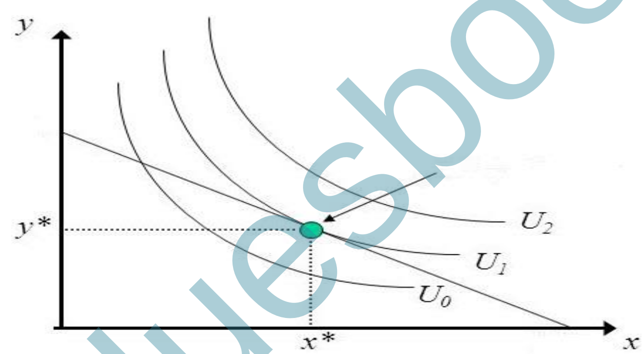
$0 = U_x\,dx + U_y\,dy \Rightarrow -\frac{U_x}{U_y} = \frac{dy}{dx}$ **(IC has –ve slope)**

$\left(\frac{P_x}{P_y}\right) = slope\ of\ budget\ line$

$$B = x\,P_x + y\,P_y$$

$y = \frac{B}{P_y} - \frac{P_x}{P_y}x \Rightarrow y = \frac{B}{P_y} + \left(-\frac{P_x}{P_y}\right)x$

$(Slope_{IC})\ \frac{U_x}{U_y} = \frac{P_x}{P_y}\ (Slope_{Budget\ Line})$



TOPIC 202: UTILITY MAXIMIZATION AND CONSUMER DEMAND: SECOND ORDER CONDITION

After developing and analyzing the 1st order condition, we develop 2nd order condition.

$U = U(x,y) + \lambda(B - xP_x + yP_y)$

$Z_{xx} = U_{xx};\ Z_{xy} = U_{xy}$

$Z_{yy} = U_{yy};\ Z_{yx} = U_{yx}$

$$|\bar{H}| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{yx} & U_{yy} \end{vmatrix}$$

$$= 2P_xP_yU_{xy} - P_y^2U_{xx} - P_x^2U_{yy} > 0$$

Diagrammatically the shape of IC should be convex to origin.

Algebraically: $\frac{d^2y}{dx^2} > 0$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{U_x}{U_y}\right) = \underbrace{-\frac{1}{U_y^2}\left(U_y\frac{dU_x}{dx} - U_x\frac{dU_y}{dx}\right)}_{Quotient\ Theorem}$$

$$\frac{dU_x(x,y)}{dx} \Rightarrow \frac{dU_x(x,y(x))}{dx}$$

$$= \frac{\partial U_x}{\partial x} + \frac{\partial U_x}{\partial y}\cdot\frac{\partial y}{\partial x}$$

$$\frac{dU_x}{dx} = U_{xx} + U_{xy}\cdot\frac{dy}{dx}$$

Similarly,

$$\frac{dU_y(x,y)}{dx} = \frac{dU_y(x,y(x))}{dx}$$

$$= \frac{\partial U_y}{\partial x} + \frac{\partial U_y}{\partial y}\cdot\frac{\partial y}{\partial x}$$

$$\frac{dU_y}{dx} = U_{yx} + U_{yy}\cdot\frac{dy}{dx}$$

$$\frac{dU_y}{dx} = \underbrace{U_{yx}}_{Young's\ Theorem} + U_{yy}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{U_x}{U_y} = -\frac{P_x}{P_y}\ \text{[F.o.C.]}$$

$$\frac{dU_x}{dx} = U_{xx} - U_{xy}\cdot\frac{P_x}{P_y};$$

$$\frac{dU_y}{dx} = U_{xy} - U_{yy}\cdot\frac{P_x}{P_y}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{U_y^2}\left(U_y\left(U_{xx} - U_{xy}\cdot\frac{P_x}{P_y}\right) - U_x\left(U_{xy} - U_{yy}\cdot\frac{P_x}{P_y}\right)\right)$$

$$= -\frac{1}{U_y^2}\left(U_yU_{xx} - U_yU_{xy}\cdot\frac{P_x}{P_y} - U_xU_{xy} + U_xU_{yy}\cdot\frac{P_x}{P_y}\right)$$

$$\frac{U_x}{U_y} = \frac{P_x}{P_y} \Rightarrow U_x = \frac{P_x}{P_y}U_y$$

$$= -\frac{1}{U_y^2}\left(U_yU_{xx} - U_yU_{xy}\cdot\frac{P_x}{P_y} - \frac{P_x}{P_y}U_yU_{xy} + \frac{P_x}{P_y}U_yU_{yy}\cdot\frac{P_x}{P_y}\right)$$

$$= -\frac{1}{U_y^2}\left(U_yU_{xx} - U_yU_{xy}\frac{P_x}{P_y} - U_yU_{xy}\frac{P_x}{P_y} + U_yU_{yy}\cdot\frac{P_x^2}{P_y^2}\right)$$

$$= -\frac{1}{U_y^2}\left(U_yU_{xx} - 2U_yU_{xy}\frac{P_x}{P_y} + U_yU_{yy}\cdot\frac{P_x^2}{P_y^2}\right)$$

$$= -\frac{1}{U_y^2}\left(\frac{U_yU_{xx}P_y^2 - 2U_yU_{xy}P_xP_y + U_yU_{yy}P_x^2}{P_y^2}\right)$$

$$= \frac{1}{U_y}\left(\frac{-U_yU_{xx}P_y^2 + 2U_yU_{xy}P_xP_y - U_yU_{yy}P_x^2}{P_y^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{2P_xP_yU_{xy} - P_y^2U_{xx} - P_x^2U_{yy}}{U_yP_y^2} = \frac{|\bar{H}|}{U_yP_y^2} > 0$$

Interpretation: Since, both, $|\bar{H}|$ and $U_yP_y^2$ are positive $\frac{d^2y}{dx^2} > 0$, fulfilling the condition for convexity (to origin).

Both S.o.C (Bordered Hessian $|\overline{H}|$) & convexity condition $\left(\frac{d^2y}{dx^2} > 0\right)$ are verified and inter-related.

TOPIC 203: NUMERICAL EXAMPLE OF UTILITY MAXIMIZATION

Considering $U(x, y) = (x+2)(y+1)$ and $P_x = 4$, $P_y = 6$ and $B = 130$

we get the budget constraint

$$P_x \cdot x + P_y \cdot y = B$$

$$4x + 6y = 130$$

Optimizing using Lagrangian function

$$\mathcal{L} = (x+2)(y+1) + \lambda(130 - 4x - 6y)$$

F.o.Cs

$$\mathcal{L}_x = \frac{\partial}{\partial x}(\mathcal{L}) = (y+1) - 4\lambda = 0 \quad \text{--- (1)}$$

$$\mathcal{L}_y = \frac{\partial}{\partial y}(\mathcal{L}) = (x+2) - 6\lambda = 0 \quad \text{--- (2)}$$

$$\mathcal{L}_\lambda = \frac{\partial}{\partial \lambda}(\mathcal{L}) = 130 - 4x - 6y = 0 \quad \text{--- (3)}$$

Solving eq. (1) & eq. (2) simultaneously.

$$\lambda = \frac{y+1}{4} \quad \& \quad \lambda = \frac{x+2}{6}$$

$$\Rightarrow \frac{y+1}{4} = \frac{x+2}{6}$$

$$6y + 6 = 4x + 8$$

$$6y = 4x + 8 - 6$$

$$6y = 4x + 2$$

$$3y = 2x + 1$$

$$y = \tfrac{1}{3}(2x+1) \quad \text{--- (4)} \Rightarrow \boxed{y^* = 11}$$

Substituting eq (4) in eq (3).

$$130 - 4x - \overset{2}{\cancel{6}}\left[\tfrac{1}{\cancel{3}}(2x+1)\right] = 0$$

$$130 - 4x - 4x - 2 = 0$$

$$128 = 8x$$

$$\boxed{x^* = 16}$$

S.o.C For constrained optimization, we resort to Bordered Hessian.

$$|\bar{H}| = \begin{vmatrix} 0 & B_x & B_y \\ B_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ B_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix}$$

$$B_x = \frac{\partial}{\partial x}(B) = \frac{\partial}{\partial x}(4x+6y) = 4$$

$$B_y = \frac{\partial}{\partial y}(B) = \frac{\partial}{\partial y}(4x+6y) = 6$$

$$\mathcal{L}_{xx} = \frac{\partial}{\partial x}(\mathcal{L}_x) = \frac{\partial}{\partial x}(y+1-4\lambda) = 0$$

$$\mathcal{L}_{yy} = \frac{\partial}{\partial y}(\mathcal{L}_y) = \frac{\partial}{\partial y}(x+2-6\lambda) = 0$$

$$\mathcal{L}_{xy} = \frac{\partial}{\partial y}(\mathcal{L}_x) = \frac{\partial}{\partial y}(y+1-4\lambda) = 1$$

$$\mathcal{L}_{yx} = \frac{\partial}{\partial x}(\mathcal{L}_y) = \frac{\partial}{\partial x}(x+2-6\lambda) = 1$$

Therefore

$$|\bar{H}| = \begin{vmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 4\begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} + 6\begin{vmatrix} 4 & 0 \\ 6 & 1 \end{vmatrix}$$

$$= 0 - 4(0-6) + 6(4-0)$$

$$= 24 + 24$$

$$= 48 > 0 \Rightarrow \text{utility is maximized}$$

$$U_{max}(16, 11) = (16+2)(11+1)$$

$$= (18)(12)$$

$$\boxed{U_{max} = 216}$$

TOPIC 204: LAW OF EQUI-MARGINAL UTILITY USING LAGRANGIAN MULTIPLIER

Consider utility function $U = Ax^a y^b$

s.t. to $P_x x + P_y y = B$

Forming Lagrangian Function.

$$\mathcal{L} = Ax^a y^b + \lambda(B - P_x x - P_y y)$$

F.o.C's

$$\mathcal{L}_x = \frac{\partial}{\partial x}(\mathcal{L}) = aAx^{a-1}y^b - \lambda P_x = 0 \quad \text{---} (1)$$

$$\mathcal{L}_y = \frac{\partial}{\partial y}(\mathcal{L}) = bAx^a y^{b-1} - \lambda P_y = 0 \quad \text{---} (2)$$

$$\mathcal{L}_\lambda = B - P_x x - P_y y = 0 \quad \text{---} (3)$$

Since $aAx^{a-1}y^{b} = MU_x$ &

$bAx^{a}y^{b-1} = MU_y$

eq. ① & ② become:

$MU_x - \lambda P_x = 0$ &

$MU_y - \lambda P_y = 0$

$\Rightarrow \lambda = \frac{MU_x}{P_x}$ & $\lambda = \frac{MU_y}{P_y}$

$$\Rightarrow \lambda = \boxed{\frac{MU_x}{P_x} = \frac{MU_y}{P_y}}$$

Which is the condition of Law of equimarginal utility

Lesson 42

HOMOGENOUS PRODUCTION FUNCTION

TOPIC 205: ECONOMIC APPLICATION OF PRODUCTION FUNCTION MAXIMIZATION

Consider a Cobb-Douglas Production Function

$$q(L,K) = 30K^{0.3}L^{0.7}$$

Budget constraint of the firm

$$5K + 4L = 100$$

Forming Lagrangian function

$$\mathcal{L} = 30K^{0.3}L^{0.7} + \lambda(100 - 5K - 4L)$$

First Order Conditions.

$$\mathcal{L}_K = 9K^{-0.7}L^{0.7} - 5\lambda = 0 \quad \text{—— (1)}$$

$$\mathcal{L}_L = 21K^{0.3}L^{-0.3} - 4\lambda = 0 \quad \text{—— (2)}$$

$$\mathcal{L}_\lambda = 100 - 5K - 4L = 0 \quad \text{—— (3)}$$

Solving eq. (1) & eq. (2) for λ

$$\lambda = \frac{9K^{-0.7}L^{0.7}}{5} \quad \& \quad \lambda = \frac{21K^{0.3}L^{-0.3}}{4}$$

Equating

$$\frac{9K^{-0.7}L^{0.7}}{5} = \frac{21K^{0.3}L^{-0.3}}{4}$$

$$\frac{L^{0.7}}{L^{-0.3}} \cdot \frac{K^{-0.7}}{K^{0.3}} = \frac{21 \times 5}{9 \times 4}$$

$$\frac{L}{K} = \frac{35}{12} \Rightarrow L = \frac{35}{12}K \quad \text{—— (4)}$$

Substituting in eq (3)

$$100 - 5K - 4\left(\frac{35}{12}\right)K = 0$$

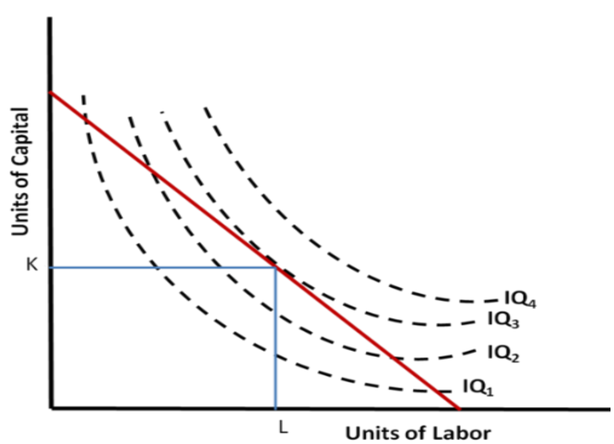
$$100 - 5K - 11.67K = 0$$

$$\boxed{K^* = 6}$$

$$L = \frac{35}{12}(6)$$

$$\boxed{L^* = 17.5}$$

$\lambda = ?$ (D.I.Y)



Maximized level of output:

$$q^*(L^*, K^*) = 30(6)^{0.3}(17.5)^{0.7}$$

$$q^*_{max} = 380.8$$

Second order Condition

Bordered Hessian

$$|H| = \begin{vmatrix} 0 & B_K & B_L \\ B_K & \mathcal{L}_{KK} & \mathcal{L}_{KL} \\ B_L & \mathcal{L}_{LK} & \mathcal{L}_{LL} \end{vmatrix}$$

$$B_K = \frac{\partial B}{\partial K} = \frac{\partial}{\partial K}(5K + 4L) = 5$$

$$B_L = \frac{\partial B}{\partial L} = \frac{\partial}{\partial L}(5K + 4L) = 4$$

$$\mathcal{L}_{KK} = \frac{\partial}{\partial K}(\mathcal{L}_K) = \frac{\partial}{\partial K}(9K^{-0.7}L^{0.7} - 5\lambda)$$

$$\mathcal{L}_{KK} = -6.3K^{-1.7}L^{0.7}$$

$$\mathcal{L}_{KL} = \frac{\partial}{\partial L}(\mathcal{L}_K) = \frac{\partial}{\partial L}(9K^{-0.7}L^{0.7} - 5\lambda)$$

$$\mathcal{L}_{KL} = 6.3K^{-0.7}L^{0.3}$$

$$\mathcal{L}_{LL} = \frac{\partial}{\partial L}(21K^{0.3}L^{-0.3} - 4\lambda)$$

$$\mathcal{L}_{LL} = -6.3K^{0.3}L^{-1.3}$$

$$\mathcal{L}_{LK} = \frac{\partial}{\partial K}(\mathcal{L}_L) = \frac{\partial}{\partial K}(21K^{0.3}L^{-0.3} - 4\lambda)$$

$$\mathcal{L}_{LK} = 6.3K^{-0.7}L^{-0.3}$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & 5 & 4 \\ 5 & -6.3K^{-1.7}L^{0.7} & 6.3K^{-0.7}L^{0.3} \\ 4 & 6.3K^{-0.7}L^{-0.3} & -6.3K^{-1.7}L^{0.7} \end{vmatrix}$$

$$|\bar{H}_3| \gtrless 0 \quad (\text{D.I.Y})$$

For output to be maximized, $|\bar{H}_3| > 0$

TOPIC 206: ECONOMIC APPLICATION ON LOGARITHMICALLY TRANSFORMED PRODUCTION FUNCTION

Consider a log-linearized Cobb-Douglas utility function.

$$U = x^{\alpha} y^{\beta} \Rightarrow \ln U = \alpha \ln|x| + \beta \ln|y|$$

Budget Constraint $B = P_x \cdot x + P_y \cdot y$

Lagrangian function

$$\mathcal{L}(x,y,\lambda) = \alpha \ln|x| + \beta \ln|y| + \lambda(B - P_x \cdot x - P_y\, y)$$

F.O.Cs.

$$\mathcal{L}_x = \frac{\partial}{\partial x}\mathcal{L}(x,y,\lambda) = \alpha\left(\frac{1}{x}\right) - \lambda P_x = 0 \quad \text{— (1)}$$

$$\mathcal{L}_y = \frac{\partial}{\partial y}\mathcal{L}(x,y,\lambda) = \beta\left(\frac{1}{y}\right) - \lambda P_y = 0 \quad \text{— (2)}$$

$$\mathcal{L}_\lambda = B - P_x \cdot x - P_y\, y = 0 \quad \text{— (3)}$$

Solving eq (1) & eq (2).

$$\lambda = \frac{\alpha}{x \cdot P_x} \quad \& \quad \lambda = \frac{\beta}{y \cdot P_y}$$

$$\Rightarrow \frac{\alpha}{x \cdot P_x} = \frac{\beta}{y \cdot P_y} \Rightarrow y = \left(\frac{\beta}{\alpha}\right)\left(\frac{P_x}{P_y}\right) x \quad \text{— (4)}$$

putting eq. (4) in eq (3)

$$B - P_x \cdot x - P_y\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{P_x}{P_y}\right)x\right\} = 0$$

$$B = x\left[P_x + \left(\frac{\beta}{\alpha}\right)P_x\right]$$

$$B = x \cdot P_x\left(1 + \frac{\beta}{\alpha}\right)$$

$$B = x \cdot P_x\left(\frac{\alpha+\beta}{\alpha}\right)$$

$$\frac{\alpha B}{P_x(\alpha+\beta)} = x$$

$$\boxed{x^* = \left(\frac{\alpha}{\alpha+\beta}\right)\frac{B}{P_x}}$$

Recalling eq (4)

$$y^* = \frac{\beta}{\alpha}\left(\frac{P_x}{P_y}\right)x^*$$

$$y^* = \frac{\beta}{\alpha}\left(\frac{P_x}{P_y}\right)\left(\frac{\alpha}{\alpha+\beta}\right)\frac{B}{P_x}$$

$$y^* = \left(\frac{\beta}{\alpha+\beta}\right)\frac{B}{P_y}$$

For a strict Cobb-Douglas function $\alpha + \beta = 1$

$$\Rightarrow \boxed{x^* = \frac{\alpha B}{P_x}} \quad \& \quad \boxed{y^* = \frac{\beta B}{P_y}}$$

TOPIC 207: HOMOGENEOUS FUNCTIONS

Etymology: Greek *homogenēs*: *homos* ‘same’ + *genos* ‘race, kind’.

Here, *genos* refers to degree of term.

A function $z = f(x, y)$ is considered homogenous if each term involved has same degree.

Polynomial of degree $\boldsymbol{n}$:

$$z = f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$$

Each term has degree $\boldsymbol{n}$.

Example: $z = f(x, y) = 3x^2y - y^3$
To check, introduce scalar (say λ) on both side.
$z' = f(\lambda x, \lambda y) = 3(\lambda x)^2(\lambda y) - (\lambda y)^3$
$= 3(\lambda^2x^2)(\lambda y) - (\lambda^3y^3)$
$= 3(\lambda^2\lambda)(x^2y) - (\lambda^3y^3)$
$= 3(\lambda^3)(x^2y) - (\lambda^3)(y^3)$
$= (\lambda^3)\{3(x^2y) - (y^3)\}$
$= (\lambda^3)\{3x^2y - y^3\}$
$z' = (\lambda^3)\{z\}$

Homogenous of degree 3 in variables x and y
D.I.Y.: Check, if functions are homogenous or not.
If yes, then of what degree.
$z = x^{0.3}.y^{0.4}$
$z = \frac{2x}{y}$

TOPIC 208: HOMOGENEOUS PRODUCTION FUNCTION, AVERAGE PRODUCTS AND CAPITAL-LABOR RATIO

Consider Cobb-Douglas production function: $Q(K, L) = AK^\alpha K^\beta$

Testing homogeneity:
$Q'(\lambda K, \lambda L) = A(\lambda K)^\alpha(\lambda K)^\beta$
$= A\lambda^\alpha K^\alpha\lambda^\beta K^\beta \Rightarrow A\lambda^\alpha\lambda^\beta K^\alpha K^\beta$
$= A\lambda^{\alpha+\beta}K^\alpha K^\beta \Rightarrow \lambda^{\alpha+\beta}(AK^\alpha K^\beta)$
$Q'(\lambda K, \lambda L) = \lambda^{\alpha+\beta}\{Q(K, L)\}$
Degree of homogeneity is $\alpha + \beta$.

For a Cobb-Douglas production function $\alpha + \beta = 1$: Linearly homogeneous or shows linear homogeneity.
Caveat: Misleading terms (linear homogeneous, linear and homogeneous)
Economic implication: Linear homogeneity implies that if all independent variables (inputs) are increased by same proportion(λ), the dependent variable (output) shall increase by same proportion⇒ Constant returns to scale.

$$(K \Rightarrow 2K \,\&\, L \Rightarrow 2L) \Rightarrow Q \Rightarrow 2Q$$

Average product of labor (AP_L):
$AP_L = \frac{Q(K,L)}{L} = \frac{AK^\alpha L^\beta}{L}$
Since, $\alpha + \beta = 1 \Rightarrow \beta = 1 - \alpha$

$$= \frac{AK^\alpha L^{1-\alpha}}{L} \Rightarrow AK^\alpha L^{-\alpha} \Rightarrow A\frac{K^\alpha}{L^\alpha} \Rightarrow A\left(\frac{K}{L}\right)^\alpha \Rightarrow Ak^\alpha$$

Relationship between average product of labor and capital-labor ratio.
Degree of homogeneity of average product of labor (AP_L):

$$AP_L = Ak^\alpha = A\left(\frac{K}{L}\right)^\alpha$$
$$AP_L(K,L) = A\left(\frac{K}{L}\right)^\alpha$$
$$AP'_L(\lambda K,\lambda L) = A\left(\frac{\lambda K}{\lambda L}\right)^\alpha$$
$$= A\left(\frac{\lambda}{\lambda}\right)^\alpha\left(\frac{K}{L}\right)^\alpha \Rightarrow A(1)^\alpha\left(\frac{K}{L}\right)^\alpha$$
$$= \lambda^0\left\{A\left(\frac{K}{L}\right)^\alpha\right\} = \lambda^0\{Ak^\alpha\}$$
$$AP'_L(\lambda K,\lambda L) = \lambda^0(AP_L)$$
$$AP_L^{Homogeneity} = 0$$

Average Product of Labor (AP_K):
$$AP_K = \frac{Q(K,L)}{K} = \frac{AK^\alpha L^{1-\alpha}}{K}$$
$$= \frac{AL^{1-\alpha}}{K\cdot K^{-\alpha}} \Rightarrow A\left(\frac{L^{1-\alpha}}{K^{1-\alpha}}\right)$$
$$= A\left(\frac{L}{K}\right)^{1-\alpha} \Rightarrow A\left(\frac{K}{L}\right)^{\alpha-1}$$
$$\Rightarrow AP_K(K,L) = Ak^{\alpha-1}$$

Relationship between average product of capital and capital-labor ratio.

Degree of homogeneity of AP_K:
$$AP_K = Ak^{\alpha-1} = A\left(\frac{K}{L}\right)^{\alpha-1}$$
$$AP_K(K,L) = A\left(\frac{K}{L}\right)^{\alpha-1}$$
$$AP'_K(\lambda K,\lambda L) = A\left(\frac{\lambda K}{\lambda L}\right)^{\alpha-1}$$
$$= A\left(\frac{\lambda}{\lambda}\right)^{\alpha-1}\left(\frac{K}{L}\right)^{\alpha-1} \Rightarrow A(1)^{\alpha-1}\left(\frac{K}{L}\right)^{\alpha-1}$$
$$= \lambda^0\left\{A\left(\frac{K}{L}\right)^{\alpha-1}\right\} = \lambda^0\{Ak^{\alpha-1}\}$$
$$AP'_K(\lambda K,\lambda L) = \lambda^0(AP_K)$$
$$AP_K^{Homogeneity} = 0$$

Interpretation of degree of homogeneity AP_K & AP_L: both are homogeneous of degree zero in the variables K and L,
Since equal proportionate changes in K and L (maintaining a constant $\frac{K}{L}$ **will not change** the magnitudes of the average products.

TOPIC 209: HOMOGENEOUS PRODUCTION FUNCTION, MARGINAL PRODUCTS AND CAPITAL-LABOR RATIO

Marginal product of capital (MP_L):
$$MP_L = \frac{\partial\{Q(K,L)\}}{\partial L} = \frac{\partial(AK^\alpha L^{1-\alpha})}{\partial L} \Rightarrow A(1-\alpha)K^\alpha L^{1-\alpha-1}$$
$$= A(1-\alpha)\frac{K^\alpha}{L^\alpha} \Rightarrow A(1-\alpha)\left(\frac{K}{L}\right)^\alpha$$
$$MP_L(K,L) = A(1-\alpha)k^\alpha$$
Relationship between marginal product of labor and capital-labor ratio.

Degree of homogeneity of MP_L:

$MP_L = A\alpha k^{\alpha-1} = A\alpha\left(\frac{K}{L}\right)^{\alpha-1}$

$MP_L(K,L) = A\alpha\left(\frac{K}{L}\right)^{\alpha-1}$

$MP'_L(\lambda K,\lambda L) = A\alpha\left(\frac{\lambda K}{\lambda L}\right)^{\alpha-1}$

$= A\alpha\left(\frac{\lambda}{\lambda}\right)^{\alpha-1}\left(\frac{K}{L}\right)^{\alpha-1}$

$= A\alpha(1)^{\alpha-1}\left(\frac{K}{L}\right)^{\alpha-1}$

$= \lambda^0\left\{A\alpha\left(\frac{K}{L}\right)^{\alpha-1}\right\} = \lambda^0\{A\alpha k^{\alpha-1}\}$

$MP'_L(\lambda K,\lambda L) = \lambda^0(MP_L)$

$$MP_L^{Homogeneity} = 0$$

Marginal product of labor (MP_K):

$MP_K = \frac{\partial\{Q(K,L)\}}{\partial K} = \frac{\partial(AK^\alpha L^\beta)}{\partial K}$

Since $\alpha + \beta = 1 \Rightarrow \beta = 1 - \alpha$

$= \frac{\partial(AK^\alpha L^{1-\alpha})}{\partial K} \Rightarrow A\alpha K^{\alpha-1}L^{1-\alpha} \Rightarrow A\alpha\frac{K^{\alpha-1}}{L^{\alpha-1}} \Rightarrow A\alpha\left(\frac{K}{L}\right)^{\alpha-1}$

$MP_K(K,L) = A\alpha k^{\alpha-1}$

Marginal product of capital and capital-labor ratio are related

Degree of homogeneity of MP_K:

$MP_K = A\alpha k^{\alpha-1} = A\alpha\left(\frac{K}{L}\right)^{\alpha-1}$

$MP_K(K,L) = A\alpha\left(\frac{K}{L}\right)^{\alpha-1}$

$MP'_K(\lambda K,\lambda L) = A\alpha\left(\frac{\lambda K}{\lambda L}\right)^{\alpha-1}$

$= A\alpha\left(\frac{\lambda}{\lambda}\right)^{\alpha-1}\left(\frac{K}{L}\right)^{\alpha-1}$

$= A\alpha(1)^{\alpha-1}\left(\frac{K}{L}\right)^{\alpha-1}$

$= \lambda^0\left\{A\alpha\left(\frac{K}{L}\right)^{\alpha-1}\right\} = \lambda^0\{A\alpha k^{\alpha-1}\}$

$MP'_K(\lambda K,\lambda L) = \lambda^0(MP_L)$

$$MP_K^{Homogeneity} = 0$$

Interpretation of degree of homogeneity MP_K & MP_L: both are homogeneous of degree zero in the variables K and L,

Since equal proportionate changes in K and L (maintaining a constant $\frac{K}{L}$ **will not change** the magnitudes of the marginal products.

TOPIC 210: HOMOGENEOUS PRODUCTION FUNCTION AND EULER'S THEOREM

Euler's theorem: Leonhard Euler, a Swiss mathematician

A property of $f(x,y)$ with degree k of homogeneous:

$$x \cdot f_1'(x,\ y) + y \cdot f_2'(x,\ y) = k \cdot f(x,\ y)$$

In production context:

$K \cdot Q'(K,\ L) + L \cdot Q'(K,\ L) = n \cdot Q(K,\ L)$

In another notation:

$K\left\{\frac{\partial Q(K,L)}{\partial K}\right\} + L\left\{\frac{\partial Q(K,L)}{\partial L}\right\} = n \cdot Q(K,\ L)$

$K\left(\frac{\partial Q}{\partial K}\right) + L\left(\frac{\partial Q}{\partial L}\right) = n \cdot Q$

For linearly homogeneous production function ($n = 1$):

$$K\left(\frac{\partial Q}{\partial K}\right) + L\left(\frac{\partial Q}{\partial L}\right) = Q$$

$K\left(\frac{\partial Q}{\partial K}\right) + L\left(\frac{\partial Q}{\partial L}\right) = Q$

$K\{A\alpha k^{\alpha-1}\} + L\{A(1-\alpha)k^{\alpha}\} = Q \quad A\alpha K k^{\alpha-1} + A(1-\alpha)Lk^{\alpha} = Q$

$A\alpha K k^{\alpha-1} + ALk^{\alpha} - A\alpha L k^{\alpha} = Q$

Restoring $k = \frac{K}{L}$

$A\alpha K\left(\frac{K}{L}\right)^{\alpha-1} + AL\left(\frac{K}{L}\right)^{\alpha} - A\alpha L\left(\frac{K}{L}\right)^{\alpha} = Q$

$A\alpha\frac{K^{\alpha}}{L^{\alpha-1}} + A\frac{K^{\alpha}}{L^{\alpha-1}} - A\alpha\frac{K^{\alpha}}{L^{\alpha-1}} = Q$

$$A\frac{K^{\alpha}}{L^{\alpha-1}} = Q \Rightarrow AK^{\alpha}L^{1-\alpha} = Q$$

Interpretation: Under CRS, if input factor is paid the amount of its ***MP***, the ***TP*** will be exhausted by the distributive shares for all the input factors.
Or, pure economic profit will be zero.
Caveat: Euler's theorem holds if perfect competition holds in factors market.

Lesson 43

COBB DOUGLAS PRODUCTION FUNCTION

TOPIC 211: COBB-DOUGLAS PRODUCTION FUNCTION AND RETURNS TO SCALE

Consider Cobb-Douglas Production Function.

$$Q(K,L) = AK^{\alpha}L^{\beta}$$

Introducing 'λ' on both sides

$$Q^{*}(\lambda K, \lambda L) = A(\lambda K)^{\alpha}(\lambda L)^{\beta}$$

$$= A\lambda^{\alpha}K^{\alpha}\cdot\lambda^{\beta}L^{\beta}$$

$$= A\lambda^{\alpha}\lambda^{\beta}K^{\alpha}L^{\beta}$$

$$= \lambda^{\alpha+\beta}\cdot(AK^{\alpha}L^{\beta})$$

$$Q^{*} = \lambda^{\alpha+\beta}(Q)$$

Since 'λ' has been completely factored out, the product function is homogenous in K and L.

However, the term $(\alpha+\beta)$ determines the nature of returns to scale.

i.e. $\alpha+\beta \gtrless 1$

$\alpha+\beta > 1 \Rightarrow$ IRS } VRS

$\alpha+\beta < 1 \Rightarrow$ DRS }

$\alpha+\beta = 1 \Rightarrow$ CRS

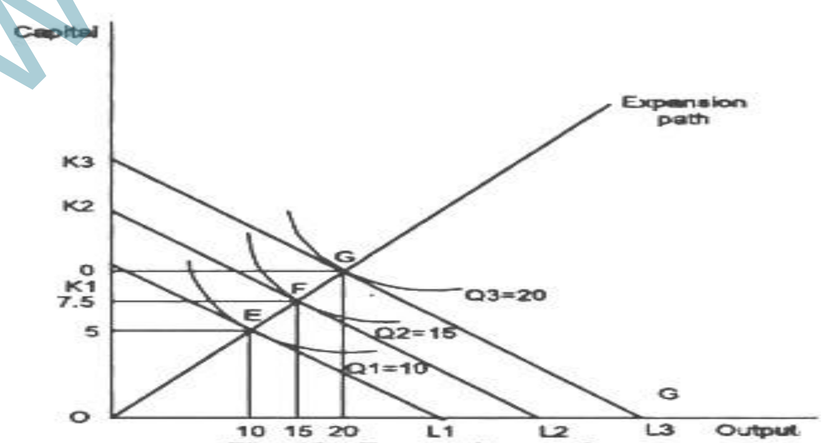


Fig. 12 : Expansion path

Numerically speaking

$Q(K,L) = 100\,K^{0.9} L^{0.1}$

$Q(\lambda K, \lambda L) = 100\,(\lambda K)^{0.9} (\lambda L)^{0.1}$

$= 100\,\lambda^{0.9} K^{0.9} \cdot \lambda^{0.1} L^{0.1}$

$= 100\,\lambda^{0.9} \lambda^{0.1} K^{0.9} L^{0.1}$

$= \lambda^{0.9} \lambda^{0.1} (100\,K^{0.9} L^{0.1})$

$Q^* = \lambda\,(Q)$

Since $(\alpha+\beta)$ exponent of λ is equal to unity, it can be inferred that current product function follows Constant Returns to Scale.

D.I.Y-1 $Q = 100\,K^{0.9} L^{0.2}$

Check homogeneity and Return to scale

D.I.Y-2 $Q = 100\,K^{0.9} L^{0.03}$

Check homogeneity & Return to scale.

TOPIC 212: HOMOGENEITY AND RETURNS TO SCALE OF THREE INPUT PRODUCTION FUNCTION

3 input production function.

$$Q(K, L, M) = AK^a L^b M^c$$

Where,

K= Capital, L= Labor, M= Material

a+b+c is the degree of homogeneity.
Introducing λ on both sides.
And Euler Theorem requires:

$$\begin{aligned} Q'(\lambda K, \lambda L, \lambda M) &= A(\lambda K)^a (\lambda L)^b (\lambda M)^c \\ &= A(\lambda)^a (K)^a (\lambda)^b (L)^b (\lambda)^c (M)^c \\ &= A(\lambda)^a\ (\lambda)^b (\lambda)^c (K)^a (L)^b\ (M)^c \\ &= A(\lambda)^{a+b+c}\ (K)^a (L)^b\ (M)^c \\ &= \lambda^{a+b+c}\{AK^a L^b M^c\} \\ &= \lambda^{a+b+c}\{Q(K, L, M)\} \end{aligned}$$

λ is completely factored out – Homogenous production function
$(a + b + c)$ is the exponent showing degree of homogeneity.
If $(a + b + c) = 1$, then CRS prevails.
If $(a + b + c) > 1$, then IRS and if $(a + b + c) < 1$, then DRS, respectively.

TOPIC 213: LEAST-COST COMBINATION IN COBB-DOUGLAS PRODUCTION FUNCTION

CES production function.

$$Q(K, L) = AK^\alpha L^\beta$$

$B = P_K.K + P_L.L$

Forming Lagrangian function:

$$\Pi(K,L,\lambda) = AK^{\alpha}L^{\beta} + \lambda(B - P_K.K - P_L.L)$$

$$\frac{\partial\{\Pi(K,L,\lambda)\}}{\partial K} = \Pi_K = \alpha AK^{\alpha-1}L^{\beta} - \lambda P_K = 0$$

$$\frac{\partial\{\Pi(K,L,\lambda)\}}{\partial L} = \Pi_L = \alpha AK^{\alpha}L^{\beta-1} - \lambda P_L = 0$$

$$\frac{\partial\{\Pi(K,L,\lambda)\}}{\partial \lambda} = \Pi_{\lambda} = B - P_K.K - P_L.L = 0$$

Solving for λ in 1st and 2nd F.O.C's and equating.

$$\frac{\alpha AK^{\alpha-1}L^{\beta}}{P_K} = \lambda$$

$$\frac{\alpha AK^{\alpha}L^{\beta-1}}{P_L} = \lambda$$

$$\frac{\alpha AK^{\alpha-1}L^{\beta}}{P_K} = \frac{\beta AK^{\alpha}L^{\beta-1}}{P_L}$$

Digressing to marginal products.

$$\frac{\partial\{Q(K,L)\}}{\partial K} = \alpha AK^{\alpha-1}L^{\beta} = MP_K$$

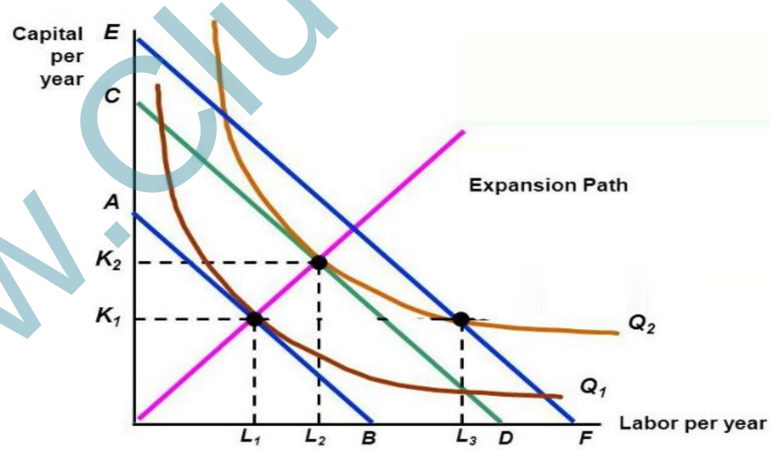
$$\frac{\partial\{Q(K,L)\}}{\partial L} = \beta AK^{\alpha}L^{\beta-1} = MP_L$$

Reverting to previous equation.

$\frac{MP_K}{P_K} = \frac{MP_L}{P_L}$ OR $\frac{P_L}{P_K} = \frac{MP_L}{MP_K}$ (Least cost input ratio).

$$\frac{P_L}{P_K} = \frac{\alpha AK^{\alpha}L^{\beta-1}}{\beta AK^{\alpha-1}L^{\beta}} = \frac{\alpha L^{-1}}{\beta K^{-1}} = \frac{\alpha K}{\beta L}$$

TOPIC 214: EXPANSION PATH USING FIRST ORDER CONDITION



Comparative-static aspect of producer equilibrium.

With fixed ratio of P_K and P_L i.e. $\overline{\left(\frac{P_K}{P_L}\right)}$, postulate successive increases of Q_o (ascent to higher and higher isoquants.

Trace the effect on the least-cost combination $\frac{K^*}{L^*}$.

Each shift of the isoquant, gives new point of tangency, with a higher Iso-cost.

Locus of such points of tangency is expansion path of the firm, serves to describe the least-cost combinations required to produce varying levels of Q_o.

1st order Condition

$$\frac{P_K}{P_L} = \frac{MP_K}{MP_L}$$

As $Q = AK^{\alpha}L^{\beta}$

$MP_K = \frac{\partial}{\partial K}(AK^{\alpha}L^{\beta}) = A\alpha K^{\alpha-1}L^{\beta}$

$MP_L = \frac{\partial}{\partial L}(AK^{\alpha}L^{\beta}) = A\beta K^{\alpha}L^{\beta-1}$

Substituting in condition.

$\frac{P_K}{P_L} = \frac{A\alpha K^{\alpha-1}L^{\beta}}{A\beta K^{\alpha}L^{\beta-1}} = \frac{\alpha K^{-1}}{\beta L^{-1}} = \frac{\alpha L}{\beta K}$

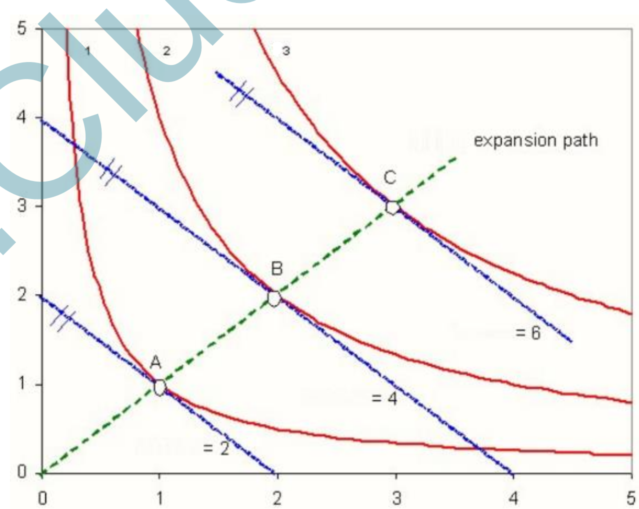
$\frac{P_K}{P_L} = \frac{\alpha L}{\beta K} \Rightarrow \frac{K}{L} = \frac{\alpha P_L}{\beta P_K}$

At equilibrium: $\frac{K^*}{L^*} = \frac{\alpha P_L}{\beta P_K}$

Answer will be numerical as $\alpha,\ \beta,\ P_K$ *and* P_L are all constants.
Therefore, all points on expansion path shall show a fixed input ratio.
Or, expansion path shall be a straight line starting from origin.
A homogeneous production function gives rise to a straight line of expansion path.

TOPIC 215: HOMOTHETIC FUNCTIONS

Etymology: Ancient Greek (*homo*-, “same”) + (*thésis*, “setting, placement, arrangement”).
Attributed Shepard (1953).
Diagrammatically speaking, homothetic functions generate radial expansions, preserving both angles and ratios of distances.



Consider a production function Q:

$$Q = f(a, b)$$

Its monotonic transformation generates a new production (composite) function H:

$H = h\{Q(a, b)\}$

Homogeneity may disappear:

$$Q = f(a,b) = a + b$$

$H = h\{Q(a,b)\} = a + b + 3.$

$Q = f(a,b)$ has a homogeneous of degree 1, however, $h\{Q(a,b)\}$ is not homogeneous.
Dependence of monotone function on original function:

$$h'(Q) \neq 0$$

Expansion path of $H(a,b)$ is linear like that $Q(a,b)$.
As observed in diagram slope of Isoquants ($MRTS_{L,K}$) will remain the same, at given (a,b):

$$Slope_{H(a,b)} = Slope_{Q(a,b)}$$

$$Slope_{H(a,b)} = -\frac{H_a}{H_b} = -\frac{\frac{\partial}{\partial a}\{H(a,b)\}}{\frac{\partial}{\partial b}\{H(a,b)\}} = -\frac{\frac{\partial}{\partial a}[h\{Q(a,\bar{b})\}]}{\frac{\partial}{\partial b}[h\{Q(\bar{a},b)\}]}$$

TOPIC 216: HOMOTHETICITY OF COBB-DOUGLAS PRODUCTION FUNCTION

Consider Cobb-Douglas production function Q:

$$Q(K,L) = AK^{\alpha}L^{\beta}$$

Its homogeneous.
Monotonic transformation (squaring) generates a new production (composite) function H:

$$H = h(Q) = Q^2$$

$$H = \left(AK^{\alpha}L^{\beta}\right)^2 = A^2K^{2\alpha}L^{2\beta}$$

Slope of Isoquant $= -\frac{H_K}{H_L}$

$$= -\frac{\frac{\partial}{\partial K}\{A^2K^{2\alpha}L^{2\beta}\}}{\frac{\partial}{\partial L}\{A^2K^{2\alpha}L^{2\beta}\}} = -\frac{A^2L^{2\beta}(2\alpha K^{2\alpha-1})}{A^2K^{2\alpha}(2\beta L^{2\beta-1})} = -\frac{\alpha L^{2\beta}(K^{2\alpha-1})}{\beta K^{2\alpha}(L^{2\beta-1})} = -\frac{\alpha L^{2\beta}(L^{-2\beta+1})}{\beta K^{2\alpha}(K^{-2\alpha+1})} = -\frac{\alpha L}{\beta K}$$

With given (K,L), slope of $H\{Q(K,L)\}$ will be constant.

$Q(K,L)$ is homogeneous of degree $(\alpha+\beta)$.

$H\{Q(K,L)\}$ is also homogeneous, but of degree $2(\alpha+\beta)$

However, a homothetic function is not necessarily homogeneous.

Lesson 44

CES PRODUCTION FUNCTION

TOPIC 217: INTRODUCING CES PRODUCTION FUNCTION

Relatively new.
Attributed to Solow, Minhas, Arrow and Chenery.
a.k.a SMAC production function
A more general form of production functions, while Cobb-Douglas can be a specific case of it.
Standard form.

$Q = A[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\nu/\rho}$

Where,

$$0 < \delta < 1,\ A > 0, -1 < \rho \neq 0, \nu > 0$$

$\delta =$ Distribution parameter
$A =$ Efficiency parameter
$\rho =$ Substitution parameter
$\nu =$ Degree of homogeneity

Generates a constant (not variable) value of elasticity of substitution (σ) between capital and labor.
Hence called constant elasticity of substitution (CES) production function.
Elasticity of substitution.

$$\sigma = \frac{1}{1+\rho}$$

$Q = 75[0.3K^{-0.4} + (0.7)L^{-0.4}]^{-1/0.4}$
$\delta = 0.3.$
$1\text{-}\delta = 0.7.$
$A = 75.$
$\rho = 0.4.$
$\nu = 1 \Rightarrow$ CRS.
As $\sigma = \frac{1}{1+\rho}$
$\sigma = \frac{1}{1+0.4}$ **= 0.71** $<$ **1** $\Rightarrow$ **less elastic –** lower substitution between capital and labor.

TOPIC 218: HOMOGENEITY OF CES PRODUCTION FUNCTION

CES production function.

$Q = A[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\nu/\rho}$

For simplicity, let $A = 1$

$$Q = [\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\nu/\rho}$$
$$Q(K, L) = [\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\nu/\rho}$$

Introducing λ on both sides.

$$Q(\lambda K, \lambda L) = [\delta(\lambda K)^{-\rho} + (1-\delta)(\lambda L)^{-\rho}]^{-\nu/\rho}$$
$$= [\delta(\lambda)^{-\rho}(K)^{-\rho} + (1-\delta)(\lambda)^{-\rho}(L)^{-\rho}]^{-\nu/\rho}$$
$$= [(\lambda)^{-\rho}[\delta(K)^{-\rho} + (1-\delta)\ (L)^{-\rho}]]^{-\nu/\rho}$$
$$= [(\lambda)^{-\rho}]^{-\nu/\rho}\ [\delta(K)^{-\rho} + (1-\delta)\ (L)^{-\rho}]^{-\nu/\rho}$$
$$= (\lambda)^{\nu}[\delta(K)^{-\rho} + (1-\delta)\ (L)^{-\rho}]^{-\nu/\rho}$$

As $\boldsymbol{\nu}$ is degree of homogeneity. CES production function is homogenous of degree $\boldsymbol{\nu}$. However, the value of $\boldsymbol{\nu}$ can be any positive number. i.e.

$\boldsymbol{\nu} \gtreqless \mathbf{1}$

If $\boldsymbol{\nu} < \mathbf{1}$, it implies decreasing returns to scale.

$$\boldsymbol{Q} = \mathbf{75}\left[\mathbf{0.3}\boldsymbol{K}^{-0.4} + (\mathbf{0.7})\boldsymbol{L}^{-0.4}\right]^{-0.5/0.4}$$

If $\boldsymbol{\nu} = \mathbf{1}$, it implies constant returns to scale.

$$\boldsymbol{Q} = \mathbf{75}\left[\mathbf{0.3}\boldsymbol{K}^{-0.4} + (\mathbf{0.7})\boldsymbol{L}^{-0.4}\right]^{-1/0.4}$$

If $\boldsymbol{\nu} > \mathbf{1}$, it implies increasing returns to scale.

$$\boldsymbol{Q} = \mathbf{75}\left[\mathbf{0.3}\boldsymbol{K}^{-0.4} + (\mathbf{0.7})\boldsymbol{L}^{-0.4}\right]^{-2/0.4}$$

TOPIC 219: MARGINAL PRODUCTS OF CES PRODUCTION FUNCTION

CES production function.

$\boldsymbol{Q} = \mathbf{75}\left[\mathbf{0.3}\boldsymbol{K}^{-0.4} + (\mathbf{0.7})\boldsymbol{L}^{-0.4}\right]^{-1/0.4}$

Where,

$$\mathbf{0} < \boldsymbol{\delta} < \mathbf{1},\ \boldsymbol{A} > \mathbf{0}, -\mathbf{1} < \boldsymbol{\rho} \neq \mathbf{0}, \boldsymbol{\nu} > \mathbf{0}$$

Marginal product of capital $\left(\boldsymbol{MP_K} = \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{K}}\right)$.

Marginal product of labor$\left(\boldsymbol{MP_L} = \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{L}}\right)$.

$$MP_K = \frac{\partial}{\partial K}\left[75\left\{0.3K^{-0.4} + 0.7L^{-0.4}\right\}^{-1/0.4}\right]$$

$$= 75\left\{\frac{\partial}{\partial K}\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-1/0.4}\right\}$$

$$= 75\left\{-\frac{1}{0.4}\right\}\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5} \cdot \frac{\partial}{\partial K}\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]$$

$$= -187.5\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5} \cdot (-0.4)(0.3)K^{-1.4}\left(\frac{\partial K}{\partial K}\right)$$

$$= -187.5(-0.12)\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5} \cdot K^{-1.4}$$

$$MP_K = 22.5K^{-1.4}\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5}$$

Assuming $K=10$, $L=10$

$$MP_L = \frac{\partial}{\partial L}\left[75\left\{0.3K^{-0.4} + 0.7L^{-0.4}\right\}^{-1/0.4}\right]$$

$$= 75\left[\frac{\partial}{\partial L}\left\{0.3K^{-0.4} + 0.7L^{-0.4}\right\}^{-1/0.4}\right]$$

$$= 75\left\{-\frac{1}{0.4}\right\}\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5}\frac{\partial}{\partial L}\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]$$

$$= -187.5\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5}(-0.4)(0.7)L^{-1.4}\left(\frac{\partial L}{\partial L}\right)$$

$$= -187.5(-0.28)\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5}L^{-1.4}$$

$$MP_L = 52.5(L^{-1.4})\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-3.5}$$

Assuming $K = 10$ & $L = 10$

Lesson 45

CES PRODUCTION FUNCTION (CONTINUED)

TOPIC 220: SHARE OF LABOR AND CAPITAL IN CES PRODUCTION FUNCTION

CES production function.

$$Q = A[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-\nu/\rho}$$

Where,

$$0 < \delta < 1,\ A > 0, -1 < \rho \neq 0, \nu > 0$$

Let $A = 1, \nu = 1$

$$Q = [\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}$$

Capital share of output $\left(S_K = \frac{(MP_K)K}{Q} = \frac{\left(\frac{\partial Q}{\partial K}\right)K}{Q}\right)$.

Labor share of output $\left(S_L = \frac{(MP_L)L}{Q} = \frac{\left(\frac{\partial Q}{\partial L}\right)L}{Q}\right)$

$$MP_K = \frac{\partial}{\partial K}\left[\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho}\right].$$

$$= \left(-\frac{1}{\rho}\right)\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho - 1}\frac{\partial}{\partial K}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}$$

$$= \left(-\frac{1}{\rho}\right)\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho - 1}(\delta)(-\rho)K^{-\rho-1}\left(\frac{\partial K}{\partial K}\right)$$

$$= \{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho}\cdot\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1}\cdot\delta(K^{-\rho-1})$$

$$= \frac{\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho}}{\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}}\cdot\frac{\delta}{K^{\rho+1}}$$

$$= \frac{Q}{[\delta K^{-\rho} + (1-\delta)L^{-\rho}]}\cdot\frac{\delta}{K^{\rho+1}}$$

$$Q = \{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho}$$

$$Q^{-\rho} = [\delta K^{-\rho} + (1-\delta)L^{-\rho}]$$

$$= \frac{Q}{Q^{-\rho}}\cdot\frac{\delta}{K^{\rho+1}}$$

$$= \frac{Q\cdot Q^{\rho}}{K^{\rho+1}}\cdot\delta = \frac{Q^{\rho+1}}{K^{\rho+1}}\cdot\delta$$

$$MP_K = \delta\left(\frac{Q}{K}\right)^{\rho+1}$$

Share of capital in output

$$S_K = \frac{MP_K\cdot K}{Q}$$

$$= \delta\left(\frac{Q}{K}\right)^{\rho+1}\cdot\frac{K}{Q}$$

$$\boxed{S_K = \delta\left(\frac{Q}{K}\right)^{\rho}}$$

$$MP_L = \frac{\partial}{\partial L}\{\delta K^{-\rho} + (1-\delta)L^{-\rho}\}^{-1/\rho}$$

As $MP_K = \delta\left(\frac{Q}{K}\right)^{\rho+1}$

Resorting virtue of symmetry

$$\Rightarrow MP_L = (1-\delta)\left(\frac{Q}{L}\right)^{\rho+1}$$

S_L = share of labour in output.

$$= \frac{MP_L \cdot L}{Q}$$

$$= (1-\delta)\left(\frac{Q}{L}\right)^{\rho+1} \cdot \frac{L}{Q}$$

$$\boxed{S_L = (1-\delta)\left(\frac{Q}{L}\right)^{\rho}}$$

Knowledge of δ, Q, L and ρ can allow the numerical values of shares of labour & capital.

TOPIC 221: CES PRODUCTION FUNCTION AND EULER'S THEOREM

CES production function.

$$Q = Ae^{\lambda t}[aK^{-\rho} + bL^{-\rho}]^{-\mu/\rho}$$

Where,

μ is the degree of homogeneity.

And Euler theorem requires:

$$\mu Y = K.Y'_K + L.Y'_L$$

Where:

$$Y'_K = \frac{\partial Q}{\partial K}$$

$$Y'_L = \frac{\partial Q}{\partial L}$$

$$Y = Ae^{\lambda t}[aK^{-\rho} + bL^{-\rho}]^{-\mu/\rho}$$

$$Y'_K \;\&\; Y'_L = \frac{\partial Y}{\partial K} \;\&\; \frac{\partial Y}{\partial L}$$

$$\frac{\partial Y}{\partial K} = \frac{\partial}{\partial K}\left[Ae^{\lambda t}\{aK^{-\rho} + bL^{-\rho}\}^{-\mu/\rho}\right]$$

$$= Ae^{\lambda t}(-\mu/\rho)[aK^{-\rho} + bL^{-\rho}]^{-\mu/\rho - 1} \cdot \frac{\partial}{\partial K}[aK^{-\rho} + bL^{-\rho}]$$

$$= Ae^{\lambda t}(-\mu/\rho)[aK^{-\rho} + bL^{-\rho}]^{-\mu/\rho - 1} \cdot (-\rho)aK^{-\rho-1}\left(\frac{\partial K}{\partial K}\right)$$

$$Y'_K = \mu Ae^{\lambda t}(a) \cdot [aK^{-\rho} + bL^{-\rho}]^{-\mu/\rho - 1} \cdot K^{-\rho-1}$$

Using virtue of Symmetry.

$$Y'_L = \frac{\partial(Y)}{\partial L} = \mu A e^{\lambda t}(b)\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho - 1} \cdot L^{-\rho-1}$$

Recalling Euler Theorem.

$$= K.Y'_K + L.Y'_L$$

$$= K\left[\mu A e^{\lambda t}(a)\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho - 1} \cdot K^{-\rho-1}\right] + L\left[\mu A e^{\lambda t}(b)\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho - 1} \cdot L^{-\rho-1}\right]$$

$$= \mu A e^{\lambda t}\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho - 1}\left\{K\, a.K^{-\rho-1} + L.bL^{-\rho-1}\right\}$$

$$= \mu A e^{\lambda t}\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho - 1}\left(aK^{-\rho} + bL^{-\rho}\right)$$

$$= \mu A\, e^{\lambda t}\, \frac{\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho}}{\left[aK^{-\rho} + bL^{-\rho}\right]} \times \left(aK^{-\rho} + bL^{-\rho}\right).$$

$$= \mu A\, e^{\lambda t}\left[aK^{-\rho} + bL^{-\rho}\right]^{-\mu/\rho}$$

$= \mu . Y \Rightarrow$ product of production function with the degree of homogeneity.

TOPIC 222: NUMERICAL CES PRODUCTION FUNCTION CALCULATION

Consider a numerical form of CES production function

$$Q = 75\left[0.3K^{-0.4} + (1-0.3)L^{-0.4}\right]^{-1/0.4}$$

and a constraint $4K + 3L = 120$.

Constrained Optimization.— Lagrangian Multiplier introduced

$$\mathcal{L} = 75\left[0.3K^{-0.4} + (0.7)L^{-0.4}\right]^{-1/0.4} +$$

$$\mathcal{L} = 75\left[0.3K^{-0.4} + 0.7L^{-0.4}\right]^{-2.5} + \lambda(120 - 4K - 3L)$$

$$\mathcal{L}_K = 22.5K^{-1.4}\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-3.5} - 4\lambda = 0 \quad \text{—— (1)}$$

$$\mathcal{L}_L = 52.5L^{-1.4}\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-3.5} - 3\lambda = 0 \quad \text{—— (2)}$$

$$\mathcal{L}_\lambda = 120 - 4K - 3L = 0 \quad \text{—— (3)}$$

Solving eq. (1) & eq. (2)

$$\frac{22.5K^{-1.4}\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-3.5}}{52.5L^{-1.4}\left(0.3K^{-0.4} + 0.7L^{-0.4}\right)^{-3.5}} = \frac{4\lambda}{3\lambda}$$

$$\frac{22.5K^{-1.4}}{52.5L^{-1.4}} = \frac{4}{3}$$

$$\Rightarrow K = 0.45L \quad \text{—— (4)}$$

Substitute in eq (3).

$$120 - 4(0.45L) - 3L = 0$$

$$L^* = 25, \quad \Rightarrow \quad K^* = 11.25$$

Maximized output of CES production function

$$Q^*(K^*, L^*) = 75\left[0.3(11.25)^{-0.4} + (0.7)(25)^{-0.4}\right]^{-2.5}$$

D.I.Y to find the maximized output Q^*.