Virtual University of Pakistan

Real Analysis I (MTH621)

Salman Amin Malik

Virtual University
Learning Management System

To my unknown students

About the instructor

Dr. Malik did his MS and PhD (Mathematics) from University of La Rochelle, La Rochelle, France in 2009 and 2012, respectively. Prior to MS and PhD, Dr. Malik completed his MPhil and MSc (Mathematics) from Department of Mathematics, University of the Punjab, Lahore, Pakistan. He has been affiliated with several universities in Pakistan and abroad. He has the experience of teaching a wide range of mathematics courses at undergraduate and graduate level.

Dr. Malik has published several research articles in international journals and conferences. His area of research includes the study of differential equations with nonlocal operators and their applications to image processing. He is also interested in inverse problems related to reaction-diffusion equations with nonlocal integro-differential operators and boundary conditions. These models have numerous applications in anomalous diffusion/transport, biomedical imaging and non-destructive testing.

About the handouts

The books followed during this course are: **W. Rudin**, Principles of Mathematical Analysis, Third Edition, McGraw-Hill, 1976. ISBN: 9780070542358. and **W. F. Trench**, Introduction to Real Analysis, Pearson Education, 2013. Consequently, the most of the examples considered in these notes are from the above mentioned books and their exercises, but not restricted to those books only. If you find any typing error in the text kindly report to me by writing an email to salman.amin.malik@gmail.com.

Course Information

Title and Course Code: Real Analysis I (MTH621)

Number of Credit Hours: 3 credits

Course Objective: The Real Analysis I is the first course towards the rigorous (formal) treatment of the fundamental concepts of mathematical analysis. This course could be considered as the fundamental course in pursue of mathematical study at undergraduate or master level. Although, the topics of the Real Analysis I are self contained but someone having knowledge of Calculus (single and multivariable) and Differential Equations will be comfortable with the contents of the course. These subjects could be considered as prerequisite of the Real Analysis I. Learning Outcomes (for the whole course) Upon completion of this course students will be able to

- Understand the set theoretic statements, the real and complex number systems.
- Apply principle of mathematical induction, ordered sets.
- Decide about the convergent or divergent sequences and series.
- Define the limit of a function, prove various theorems about limits, sequences and functions.
- Check the continuity of real valued functions, prove various theorems about continuous functions with emphasize on the proofs.
- Understand the derivative of a function, proof of various theorems about differentiability of the function (LO6).
- Prove and apply Bolzano-Weierstrass theorem, Mean value theorem.
- Define Riemann integral, Riemann sums, proof of various results about the Riemann integrals.

Prerequisites: Calculus with Analytical Geometry

The textbooks for this course:

[1] W. Rudin, Principles of Mathematical Analysis, Third Edition, McGraw-Hill, 1976. ISBN: 9780070542358.

[2] W. F. Trench, Introduction to Real Analysis, Pearson Education, 2013.

Reference books:

[3] A. N. Kolmogorov and S. V. Fomin, Introductory Real Analysis, Revised English Edition Translated and Edited by R. A. Silverman, Dover Publication, Inc. New York.

[4] R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, Third Edition, 2000, John Wiley & Sons Inc.

- The real number system
- Sequences and Series
- Limits, Continuity and Differentiability
- The integration

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Chapter 1

The Real Number System

1.1 Basic Set Theory

In this section, we are going to provide some basic terminology required to understand the forthcoming concepts about the real number system.

1.1.1 Universal Set

The understanding that the members of all sets under consideration in any given context come from a specific collection of elements, called the **universal set**.

For this chapter of the course our universal set will be the set of real numbers.

If an element x is in A, we write $x \in A$ and say that x is a member of A. If an element x is not in A, we write $x \notin A$ and say that x is not a member of A.

How to define a set?

$$\{4, 5, 6, 7\}.$$

1.1.2 Set Builder Notion

If P is a property that is meaningful and unambiguous for elements of a set S, then we write

$$\{x \in S : \; P(x)\},$$

for the set of all elements x in S for which the property P is true.

If every element of a set A also belongs to a set B, we say that A is a subset of B and write $A \subset B$.

Proper subset: We say that A is proper subset of B if there exist at least one element of B which is not in A.

Equal sets: Two sets are said to be equal if they contain the same number of elements.

Examples: Consider the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$.

- $\{x \in \mathbb{N} : x^2 - 3x + 2 = 0\} = \{1, 2\}$.

- $\{x \in \mathbb{N} : x^2 - 4x + 2 = 0\}$.

- The set of even numbers
$$\{2k : \ k \in \mathbb{N}\}.$$

- The set of odd numbers
$$\{2k - 1 : \ k \in \mathbb{N}\}.$$

Let S and T be sets.

- S *contains* T, and we write $S \supset T$ or $T \subset S$, if every member of T is also in S. In this case, T is a *subset* of S.

- $S - T$ is the set of elements that are in S but not in T.

- S *equals* T, and we write $S = T$, if S contains T and T contains S; thus, $S = T$ if and only if S and T have the same members.

Let S and T be sets.

- S *strictly contains* T if S contains T but T does not contain S; that is, if every member of T is also in S, but at least one member of S is not in T.

- The *complement* of S, denoted by S^c, is the set of elements in the universal set that are not in S.

- The *union* of S and T, denoted by $S \cup T$, is the set of elements in at least one of S and T

Let S and T be sets.

- The *intersection* of S and T, denoted by $S \cap T$, is the set of elements in both S and T If $S \cap T = \emptyset$ (the empty set), then S and T are *disjoint sets*

- A set with only one member x_0 is a *singleton set*, denoted by $\{x_0\}$.

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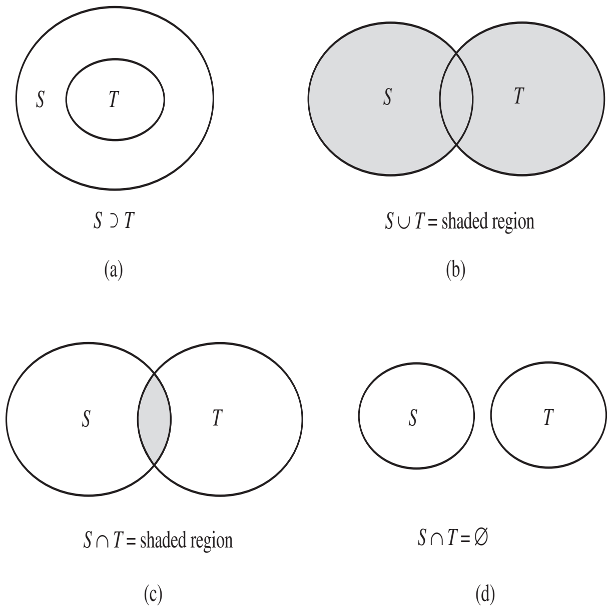


Figure 1.1: (a) Subset (b) Union of two sets (c) Intersection of two sets, (d) Disjoint sets

- **The set of natural numbers**
- **The set of prime numbers**
- **Fundamental Theorem of arithmetic**
- **Diophantines**
- **A little bit about number theory**

The set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, ...\}.$$

1.2 Number Theory

Number theory is a branch of mathematics that studies the properties of, and the relationships between, particular types of numbers.

- The set of natural numbers $\mathbb{N}$.
- The prime numbers.

The primes are the building blocks of the positive integers.
Fundamental Theorem of Arithmetic: Every positive integer can be uniquely written as the product of primes in nondecreasing order.

How may prime numbers are there? (2500 years ago, Euclid provided the proof)

Many different approaches have been used to determine whether an integer is prime. **For example, in the nineteenth century, Pierre de Fermat showed that p divides $2^p - 2$ whenever p is prime**.
The search for integer solutions of equations is another important part of number theory.
An equation with the added provision that only integer solutions are sought is called **diophantine**, after the ancient Greek mathematician Diophantus.
Example:

$$a^n + b^n = c^n, \qquad n \in \mathbb{Z}.$$

The set of nonnegative integers:

$$\mathbb{Z}^+ = \{0, 1, 2, 3, ...\}.$$

The set of nonpositive integers:

$$\mathbb{Z}^- = \{0, -1, -2, -3, ...\}.$$

The set of integers:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}.$$

Some properties of the natural numbers:

$$2 + 4 = 4 + 2$$

$$m + 3 = p + 3 \quad \text{then} \quad m = p.$$

$$4 + (2 + 7) = (4 + 2) + 7.$$

1.3 Principle of Mathematical Induction

Let $P(n)$be a mathematical statement, where $n \in \mathbb{N}$ (or $\mathbb{Z}^+$). If

- $P(1)$ is true.

- $P(n)$ is true implies $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$ (or $\mathbb{Z}^+$).

Examples:

- $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.
- $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$.
- Prove that if $(x + 1/x)$ is an integer then $(x^n + 1/x^n)$ is also an integer for any positive integer n.

Theorem: Suppose that $m, n, p \in \mathbb{Z}^+$. Then

- $m + n = n + m$, (commutativity)
- $(m + n) + p = n + (m + p)$, (associativity)
- if $m + n = p + n$, then $m = p$ (cancelation)
- if $m + n = 0$, then $m = n = 0$

Proof: Proof of first property.
Step I: Define $U = \{m \in \mathbb{Z}^+ : 0 + m = m + 0\}$.

Step II: Define $V = \{n \in \mathbb{Z}^+ : (m + 1) + n = (m + n) + 1, \text{ for all } m \in \mathbb{Z}^+\}$.

Step III: Define $W = \{n \in \mathbb{Z}^+ : m + n = n + m, \text{ for all } m \in \mathbb{Z}^+\}$.

$$m + (n + 1) = (m + n) + 1 \tag{1.1}$$

Step I: Define $U = \{m \in \mathbb{Z}^+ : 0 + m = m + 0\}$. Notice that $0 \in U$. Suppose $m \in U$ and consider $0 + (m + 1) = (m + 1)$ and we have $(0 + m) + 1 = m + 1$.

Thus $m + 1 \in U$, and so $U = \mathbb{Z}^+$.
Step II: Define $V = \{n \in \mathbb{Z}^+ : (m + 1) + n = (m + n) + 1, \text{ for all } m \in \mathbb{Z}^+\}$.
$0 \in V$, suppose $n \in V$, then

$$\begin{aligned} (m+1) + (n+1) &= ((m+1)+n)+1 \quad \text{by} \quad (1.1) \\ &= ((m+n)+1)+1, \quad \text{since} \quad n \in V \\ &= (m+(n+1))+1 \quad \text{by} \quad (1.1). \end{aligned}$$

Hence we have $V = \mathbb{Z}^+$.
Step III: Define $W = \{n \in \mathbb{Z}^+ : m + n = n + m, \text{ for all } m \in \mathbb{Z}^+\}$.
Using the first step we have $0 \in W$. Suppose $n \in W$. Then

$$\begin{aligned} m + (n+1) &= (m+n)+1 \quad \text{by } (1.1) \\ &= (n+m)+1, \quad \text{since} \quad n \in W \\ &= (n+1)+m, \quad \text{from step II.} \end{aligned}$$

Hence $W = \mathbb{Z}^+$.

Theorem: Suppose that $m, n, p \in \mathbb{Z}^+$. Then

- $m.n = n.m$, (commutativity)
- $(m.n).p = n.(m.p)$, (associativity)
- if $m.n = p.n$ and $n \neq 0$, then $m = p$ (cancellation)
- if $m.n = 0$, then $m = 0$ or $n = 0$.
- $0.n = 0$ and $1.n = n$.

1.3.1 An initial segment

An initial segment I of $\mathbb{N}$ is a nonempty subset of $\mathbb{N}$ with the property that if $n \in I$ and $m \leq n$ then $m \in I$.

Example:

- Consider the set
$$\{1, 2, 3, ..., 20\}.$$
- Consider the set
$$\{1, 2, 3, ..., n\}.$$
- Consider the set
$$\{2, 4, 6, 8, 10\}.$$

Proposition: If I is an initial segment of $\mathbb{N}$ then either $I = \mathbb{N}$ or there exists $n \in \mathbb{N}$ such that $I = I_n = \{m \in \mathbb{N} : m \leq n\}$.

Proof: It follows immediately from the definition of an initial segment that if $m \notin I$ and $n \geq m$ then $n \notin I$. If $I \neq \mathbb{N}$, then $\mathbb{N} \backslash I$ is non-empty. Let m_0 be its least element. Suppose, if possible, that $m_0 = 1$. If $n \in \mathbb{N}$, then $n \geq 1$, so that $n \notin I$ and I is the empty set. Thus $m_0 > 1$, and so there exists $n \in \mathbb{N}$ such that $m_0 = n + 1$. Then $n \in I$, and so $I_n \subseteq I$. But if $p > n$ then $p \geq n + 1 = m_0$, and so $p \notin I$. Thus $I \subseteq I_n$.

Recall a one to one and onto mapping: A mapping (function) $f : A \to B$ is said to be one to one if
$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

A mapping (function) $f : A \to B$ is said to be onto if for every $y \in B$ there exists at least an $x \in A$ such that
$$f(x) = y.$$

A mapping (function) is said to be bijective if the mapping is one to one and onto.

1.4 Finite and Infinite Set

A set A is finite if either A is empty or there exists $n \in \mathbb{N}$ and a bijective mapping $c : I_n \to A$. Thus the finite sequence $(c_1, ..., c_n)$ lists the elements of A, without repetition.

A set is infinite if it is not finite.

Proposition: If $g : I_m \to I_n$ is an one to one (injective) mapping then $m \leq n$.

Proof:

- The proof is by induction on m. The result is trivially true if $m = 1$.
- Suppose that it holds for m, and that $f : I_{m+1} \to I_n$ is injective. Then $m + 1 > 2$, so that $f(I_{m+1})$ contains at least two points, and so $n = k + 1$, for some $k \in \mathbb{N}$.
- Let $\tau : I_n \to I_n$ be the mapping that transposes $f(m+1)$ and n and leaves the other elements of I_n fixed.
- Then $\tau \circ f : I_{m+1} \to I_n$ is injective, and $\tau(f(I_m)) \subseteq I_k$.
- By the inductive hypothesis, $m \leq k$, and so $m + 1 \leq k + 1 = n$.

Corollary: If A is a non-empty finite set, there exists a unique $n \in \mathbb{N}$ for which there exists a bijection $c : I_n \to A$.

Proof: Suppose that $c : I_n \to A$ and $c' : I_{n'} \to A$ are bijections.
Then $c^{-1} \circ c' : I_{n'} \to I_n$ is a bijection, and so $n' \leq n$.

Similarly, $n \leq n'$.

Remark: The number n is known as cardinality of A and is written as $\mid A \mid$.

Proposition: Suppose that A is a finite set, and that $f : A \to B$ is a bijection. Then B is finite, and $\mid B \mid = \mid A \mid$.

Proof: For if $C : I_{|A|} \to A$ is a bijection, then the mapping

$$f \circ C : I_{|A|} \to B$$

is the bijection.

1.5 Field

A field is a set F, together with two laws of composition, addition $(+)$ and multiplication $(.)$ such that for all $a, b, c \in F$ the following properties holds

- $a + b = b + a$ and $ab = ba$ (commutative laws).
- $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (associative laws).
- $a(b + c) = ab + ac$ (distributive law).
- There are distinct members 0 and 1 such that $a + 0 = a$ and $a1 = a$ for all a.
- For each $a \in F$ there is an element $-a \in F$ such that $a + (-a) = 0$, and if $a \neq 0$, there is an element $1/a$ such that $a(1/a) = 1$.

Remark: The left distributive law also holds.
Examples:

- Let $z_2 = \{0, 1\}$ such that

  $$0 + 0 = 1 + 1 = 0; 0 + 1 = 1 + 0 = 1$$

  and

  $$0.0 = 0.1 = 1.0 = 0, 1.1 = 1.$$

- Is the set of integers a field?

1.6 Dedekind Infinite Set

Dedekind defined a set A to be infinite if there is an injective map $j : A \to A$ which is not onto (surjective); such sets are now called Dedekind infinite.

Example: Show that $\mathbb{N}$ is Dedekind infinite.
The mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is injective, and is not surjective.

Corollary: The set of natural numbers $\mathbb{N}$ is infinite set.

Countable sets: A set A is countable if it is finite or if there is a bijection $c : \mathbb{N} \to A$; otherwise it is uncountable.

Remark: A set is countable if it is empty or if there is a bijection from an initial segment of $\mathbb{N}$ onto A. The function c is called an enumeration of A. A set is countably infinite if it is infinite and countable.

Thus A is countably infinite if and only if the elements of A can be listed, or enumerated, as an infinite sequence $(c_1, c_2, ...)$, without repetition.

If A is countable (countably infinite) and $j : A \to B$ is a bijection, then B is countable (countably infinite). Not every set is countable. It was Cantor who first showed, in 1873, that there are different sizes of infinite set, showing that the set of real numbers is uncountable.

Proposition: Let A be a non-empty set. Then the following are equivalent.

(a) A is countable.

(b) There exists a surjection $f : \mathbb{N} \to A$.

(c) There exists an injection $g : A \to \mathbb{N}$.

Theorem: The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof: Define the mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f(k, l) = 2^k 2^l.$$

Use above proposition to prove that $\mathbb{N} \times \mathbb{N}$ is countable.

1.7 The Set of Rational Numbers

Construction of set of rational numbers: Let $\mathbb{Z}^* = \mathbb{Z} \backslash \{0\}$ be the set of nonzero integers. Define a relation on $\mathbb{Z} \times \mathbb{Z}^*$ by setting $(p, q) \sim (r, s)$ if $ps = qr$.

Proposition: The relation $(p, q) \sim (r, s)$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$.
Proof:
Transitive: Suppose $(p, q) \sim (r, s)$ and $(r, s) \sim (t, u)$, we need to show that $(p, q) \sim (t, u)$. Consider

$$pusr = (ps)(ru) = (qr)(ts) = qtsr$$

$\mathbb{Q}$ **is abelian group under addition**: We define addition on $\mathbb{Q}$ by

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}.$$

We define the multiplication as $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right) = \frac{pr}{qs}$.

Proposition: Let $\mathbb{Q}^* = \mathbb{Q}/\{0/1\}$. Then, $(\mathbb{Q}^*, .)$ and $(\mathbb{Q}^*, +)$ are abelian group.

The set of rational numbers is defined as

$$\mathbb{Q} = \{\frac{m}{n} \;:\; m, n \in \mathbb{Z}, n \neq 0\}.$$

Example: Find solution of the equation $p = \sqrt{2}$ in the set of rational numbers if possible.

Solution: See Lecture.

Example: Let $A = \{p \in \mathbb{Q} : \; p^2 < 2\}$. We will show that for every $p \in A$, we can find a rational number q such that $p < q$.

To do this, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1.2}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{1.3}$$

- If p is in A then $p^2 - 2 < 0$, (1.2) shows that $q > p$ and (1.3) shows that $q^2 < 2$. Thus q is in A.

Let $B = \{p \in \mathbb{Q} : \; p^2 > 2\}$. We will show that for every $p \in B$, we can find a rational number q such that $q < p$.

- If p is in B then $p^2 - 2 > 0$, (1.2) shows that $0 < q < p$ and (1.3) shows that $q^2 > 2$. Thus q is in B.

1.8 Ordered Set

Let S be a set. An order on S is a relation , denoted by $<$, with the following properties

- For each pair $a, b \in S$, exactly one of the following is true:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

- If $a, b, c \in S$ such that If $a < b$ and $b < c$, then $a < c$. (The relation $<$ is *transitive*.)

- If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $ac < bc$.

Example: The set of rational numbers $\mathbb{Q}$ is an order set, if $r < s$ is defined to means that $s - r$ is a positive rational number.

An ordered set is a set S in which an order is defined.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which one of these two is hold.

What is the negation of $x > y$?

1.8.1 Partial Order

: A partial order is a relation R on a set S such that for all $a, b, c \in S$, we have the following

- aRa (Reflexivity).
- If aRb and bRa implies $a = b$ (Anti Symmetric).
- If aRb, bRc then aRc (Transitve).

1.9 Lower Bound

Suppose the set S is an ordered set, and $E \subset S$. If there exists a b such that $x \geq b$ whenever $x \in E$. We say that E is bounded below. In this case, b is a lower bound of E.

If b is a lower bound of E, then so is any smaller number.

1.9.1 Infimum

If α is a lower bound of E, but no number greater than α is a lower bound of E, then α is an infimum of E, and we write

$$\alpha = \inf E.$$

Example: Recall the set $A = \{p \in \mathbb{Q} : \ p^2 > 2\}$.

1.10 Upper Bound

Suppose the set S is an ordered set, and $E \subset S$. If there exists a b such that $x \leq b$ whenever $x \in E$. We say that E is bounded above In this case, b is an upper bound of E.

If b is an upper bound of E, then so is any larger number

1.10.1 Supremum

If β is an upper bound of E, but no number less than β is an upper bound of E, then β is a supremum of E, and we write

$$\beta = \sup E.$$

Example: Recall the set $A = \{p \in \mathbb{Q} : \ p^2 < 2\}$.

Example: If S is the set of negative numbers, then any nonnegative number is an upper bound of S, and $\sup S = 0$.

Example: If S_1 is the set of negative integers, then any number a such that $a \geq -1$ is an upper bound of S_1, and $\sup S_1 = -1$.

Remark: The supremum or infimum of a set may or may not belong to that set.

Example: Consider the set

$$E = \{p \in \mathbb{Q} \ : p < 0\}.$$

What are the upper bounds?
What is the supremum of this set?
Is it belong to the set?

Example: Consider the set

$$E_1 = \{p \in \mathbb{Q} \ : p \leq 0\}.$$

What are the upper bounds?
What is the supremum of this set?
Is it belong to the set?

Example: Consider the set

$$E_2 = \{1/n \ : n \in \mathbb{N}\}.$$

1.11 Least Upper Bound Property

An ordered set S is said to have the least-upper-bound property if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then sup E exists in S.
Recall the set $A = \{p \in \mathbb{Q} : \ p^2 > 2\}$.

Recall the set $B = \{p \in \mathbb{Q} : \ p^2 < 2\}$.

Consider the set

$$E_2 = \{1/n \ : n \in \mathbb{N}\}.$$

Theorem: Suppose S is an ordered set with the least-upper-bound property, $B \subseteq S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$a = \sup L$$

exists in S, and $a = \inf B$. In particular $\inf B$ exists in S.

Proof: L is not empty set.

L is bounded above.

S satisfy least-upper-bound property therefore L has a supremum in S; call it a.

If $\gamma < a$ then γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $a \leq x$ for every $x \in B$.

Thus $a \in L$.

If $a < \beta$ then $\beta \notin L$, since a is an upper bound of L.

We have shown that $a \in L$ but $\beta \notin L$ if $\beta > a$.

In other words, a is a lower bound of B, but β is not if $\beta > a$.

This means that $a = \inf B$.

Theorem: There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property

Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

1.12 The Completeness Axiom

If a nonempty set of real numbers is bounded above, then it has a supremum.

The above property is called completeness, and we say that the real number system is a complete ordered field.

Theorem: If a nonempty set S of real numbers is bounded above, then $\sup S$ is the unique real number β such that

- $x \leq \beta$ for all x in S;
- if $\varepsilon > 0$ (no matter how small), there is an x_0 in S such that $x_0 > \beta - \varepsilon$.

Proof: We first show that $\beta = \sup S$ has first and second properties. Since β is an upper bound of S, it must satisfy the first property. Since any real number a less than β can be written as $\beta - \varepsilon$ with $\varepsilon = \beta - a > 0$, second property is just another way of saying that no number less than β is an upper bound of S. Hence, $\beta = \sup S$ satisfies first and second properties.

Now we show that there cannot be more than one real number with first and second properties. Suppose that $\beta_1 < \beta_2$ and β_2 has the second property ; thus, if $\varepsilon > 0$, there is an x_0 in S such that $x_0 > \beta_2 - \varepsilon$. Then, by taking $\varepsilon = \beta_2 - \beta_1$, we see that there is an x_0 in S such that

$$x_0 > \beta_2 - (\beta_2 - \beta_1) = \beta_1,$$

so β_1 cannot have the first property. Therefore, there cannot be more than one real number that satisfies both parts.

1.13 The Archimedean Property of $\mathbb{R}$

Theorem: If ρ and ε are positive, then $n\varepsilon > \rho$ for some integer n.

Proof: The proof is by contradiction. If the statement is false, ρ is an upper bound of the set

$$S = \{x = n\varepsilon,\ n \in \mathbb{Z}\}.$$

Therefore, S has a supremum β (Why?), by definition of least upper bound property of real numbers.

Therefore,

$$n\varepsilon \leq \beta \quad \text{for all integers } n. \tag{1.4}$$

Since $n+1$ is an integer whenever n is, (1.4) implies that

$$(n+1)\varepsilon \leq \beta$$

and therefore

$$n\varepsilon \leq \beta - \varepsilon$$

for all integers n.

1.14 Dense Set in $\mathbb{R}$

A set D is said to be dense in the set of real numbers if every open interval (a, b) contains a member of D.

Theorem: The rational numbers are dense in $\mathbb{R}$, that is, if a and b are real numbers with $a < b$, there is a rational number p/q such that $a < p/q < b$.

Recall the The Archimedean property

Theorem: If ρ and ε are positive, then $n\varepsilon > \rho$ for some integer n.
Proof of the theorem: The Archimedean property with $\rho = 1$ and $\varepsilon = b - a$, there is a positive integer q such that $q(b-a) > 1$.

There is also an integer j such that $j > qa$. This is obvious if $a \leq 0$, and it follows from Archimedean property with $\varepsilon = 1$ and $\rho = qa$ if $a > 0$.

Let p be the smallest integer such that $p > qa$. Then $p - 1 \leq qa$, so

$$qa < p \leq qa + 1.$$

Since $1 < q(b-a)$, this implies that

$$qa < p < qa + q(b-a) = qb,$$

so $qa < p < qb$. Therefore, $a < p/q < b$.

1.15 The Set of Rational Numbers is not Complete

The rational number system is not complete; that is, a set of rational numbers may be bounded above (by rational numbers), but not have a rational upper bound less than any other rational upper bound, that is, that set does not have a rational supremum.

Recall: **Theorem**: If a nonempty set S of real numbers is bounded above, then $\sup S$ is the unique real number β such that

- $x \leq \beta$ for all x in S;
- if $\varepsilon > 0$ (no matter how small), there is an x_0 in S such that $x_0 > \beta - \varepsilon$.

Consider the set

$$A = \{p \in \mathbb{Q} : \ p^2 < 2\}.$$

If $p \in A$, then $p < \sqrt{2}$.

Then using the fact that there is a rational number between every two real numbers implies that if $\varepsilon > 0$ there is a rational number r_0 such that $\sqrt{2} - \varepsilon < r_0 < \sqrt{2}$, so using above Theorem implies that $\sqrt{2} = \sup A$. However, $\sqrt{2}$ is *irrational*; that is, it cannot be written as the ratio of integers.

Therefore, if r_1 is any rational upper bound of A, then $\sqrt{2} < r_1$. Since there is a rational number between every two real numbers, there is a rational number r_2 such that $\sqrt{2} < r_2 < r_1$. Since r_2 is also a rational upper bound of A, this shows that A has no rational supremum.

ExampleProduct of a rational and irrational number is irrational.

Solution: See lecture.

Example: Sum of a rational and an irrational number is irrational.

Proof: See lecture.

Theorem: The set of irrational numbers is dense in the reals; that is, if a and b are real numbers with $a < b$, there is an irrational number t such that $a < t < b$.

Proof: Since between every two real numbers there is a rational number, therefore there are rational numbers r_1 and r_2 such that

$$a < r_1 < r_2 < b. \tag{1.5}$$

Let

$$t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1).$$

Then t is irrational (why?) and $r_1 < t < r_2$, so $a < t < b$, from (1.5).

1.16 Applications of Properties of Field

Proposition: The axioms of addition imply the following

(a) If $x + y = x + z$ then $y = z$ (**Cancellation law**).

(b) If $x + y = x$ then $y = 0$ (**Uniqueness of the additive identity**).

(c) If $x + y = 0$ then $y = -x$ (**Uniqueness of the additive inverse**).

(d) $-(-x) = x$.

Proof: If $x + y = x + z$, then from the axioms of field we have

$$\begin{aligned} y = 0 + y &= (-x + x) + y = -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

This proves (a).

- Take $z = 0$ in (a) to obtain (b).
- Take $z = -x$ in (a) to obtain (c).
- Since $-x + x = 0$, (c) with $-x$in place of x gives (d).

Proposition: The axioms of multiplication imply the following

(a) If $x \neq 0$ and $xy = xz$ then $y = z$ (**Cancellation law**).

(b) If $x \neq 0$ and $xy = x$ then $y = 1$ (**Uniqueness of the multiplicative identity**).

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$ (**Uniqueness of the additive inverse**).

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proposition: The field axioms imply the following statements for any $x, y, z \in F$

(a) $0x = 0$.

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

The manipulative properties of the real numbers, such as the relations

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2, \\ (3a+2b)(4c+2d) &= 12ac + 6ad + 8bc + 4bd, \\ (-a) &= (-1)a, \quad a(-b) = (-a)b = -ab, \\ \frac{a}{b} + \frac{c}{d} &= \frac{ad+bc}{bd} \quad (b, d \neq 0), \end{aligned}$$

all follow from the axioms of field.

Theorem: For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.

Proof: The uniqueness is clear, since if there are two positive numbers y_1 and y_2 then

$$0 < y_1 < y_2 \quad \Rightarrow \quad y_1^n < y_2^n.$$

Let E be the set containing of all positive real numbers t such that $t^n < x$.

The set E is not empty, as if $t = x/(x+1)$ then $0 \leq t < 1$ and $t^n < t < x$.

If $t > 1 + x$ then $t^n \geq t > x$, so that $t \notin E$. Thus $t \in E$ is an upper bound of E.

Then by least upper bound property there exists $y \in \mathbb{R}$ such that

$$y = \sup E.$$

We need to prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction.

Recall the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + ... + a^{n-1})$ yields the identity

$$b^n - a^n = (b-a)nb^{n-1},$$

when $0 < a < b$.

Assume $y^n < x$, choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Put $a = y$, $b = y + h$. Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x$, and $y + h \in E$, which is contradiction to the fact that y is an upper bound of E.

Assume $y^n > x$, put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, we conclude that

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x$, and $t \notin E$. It follows that $y - k$ is an upper bound of E. But $y - k < y$, which contradicts the fact that y is the least upper bound of E.

Theorem: If a and b are any two real numbers, then

$$|a + b| \leq |a| + |b|. \tag{1.6}$$

Proof: There are four possibilities:

(a) If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$, so $|a + b| = a + b = |a| + |b|$.

(b) If $a \leq 0$ and $b \leq 0$, then $a + b \leq 0$, so $|a + b| = -a + (-b) = |a| + |b|$.

(c) If $a \geq 0$ and $b \leq 0$, then $a + b = |a| - |b|$.

(d) If $a \leq 0$ and $b \geq 0$, then $a + b = -|a| + |b|$.

Eq. 1.6 holds in cases **(c)** and **(d)**, since

$$|a + b| = \begin{cases} |a| - |b| & \text{if } |a| \geq |b|, \\ |b| - |a| & \text{if } |b| \geq |a|. \end{cases} \tag{1.7}$$

Corollary: If a and b are any two real numbers, then

$$|a - b| \geq \big||a| - |b|\big|, \qquad \text{and} \qquad |a + b| \geq \big||a| - |b|\big|. \tag{1.8}$$

Proof: Replacing a by $a - b$ in (1.6) yields

$$|a| \leq |a - b| + |b| \quad \Rightarrow \quad |a - b| \geq |a| - |b|.$$

Interchanging a and b here yields

$$|b - a| \geq |b| - |a|,$$

which is equivalent to

$$|a - b| \geq |b| - |a|, \tag{1.9}$$

since $|b - a| = |a - b|$. Since

$$\big||a| - |b|\big| = \begin{cases} |a| - |b| & \text{if} \quad |a| > |b|, \\ |b| - |a| & \text{if} \quad |b| > |a|. \end{cases}$$

(**??**) and (1.9) imply (1.8). Replacing b by $-b$ in (1.8) yields (**??**), since $|-b| = |b|$.

1.17 The Extended Real Number System

A nonempty set S of real numbers is *unbounded above* if it has no upper bound, or *unbounded below* if it has no lower bound.

It is convenient to adjoin to the real number system two fictitious points, $+\infty$ (which we usually write more simply as ∞) and $-\infty$, and to

Define the order relationships between them and any real number x by

$$-\infty < x < \infty. \tag{1.10}$$

We call ∞ and $-\infty$ *points at infinity*.

If S is a nonempty set of reals, we write

$$\sup S = \infty \tag{1.11}$$

to indicate that S is unbounded above, and

$$\inf S = -\infty \tag{1.12}$$

to indicate that S is unbounded below.

$$\text{If} \quad S = \{x \;:\; x < 2\},$$

then $\sup S = 2$ and $\inf S = -\infty$.

$$\text{If} \quad S = \{x \;:\; x \geq -2\},$$

then $\sup S = \infty$ and $\inf S = -2$.

If S is the set of all integers, then $\sup S = \infty$ and $\inf S = -\infty$.

A member of the extended reals differing from $-\infty$ and ∞ is *finite*; that is, an ordinary real number is finite. However, the word "finite" in "finite real number" is redundant and used only for emphasis, since we would never refer to ∞ or $-\infty$ as real numbers. The real number system with ∞ and $-\infty$ adjoined is called the *extended real number system*, or simply the *extended reals*.

We must defined arithmetic operations with $\pm\infty$. A member of the extended reals differing from $-\infty$ and ∞ is *finite*; that is, an ordinary real number is finite. However, the word "finite" in "finite real number" is redundant and used only for emphasis, since we would never refer to ∞ or $-\infty$ as real numbers.

The arithmetic relationships among ∞, $-\infty$, and the real numbers are defined as follows.

- If a is any real number, then

$$\begin{aligned} a + \infty &= \infty + a = \infty, \\ a - \infty &= -\infty + a = -\infty, \\ \frac{a}{\infty} &= \frac{a}{-\infty} = 0. \end{aligned}$$

- If $a > 0$, then

$$\begin{aligned} a\,\infty &= \infty a = \infty, \\ a\,(-\infty) &= (-\infty)\,a = -\infty. \end{aligned}$$

- If $a < 0$, then

$$\begin{aligned} a\infty &= \infty a = -\infty, \\ a(-\infty) &= (-\infty)a = \infty. \end{aligned}$$

We also define

$$\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$$

and

$$-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty.$$

Finally, we define

$$|\infty| = |-\infty| = \infty.$$

It is not useful to define $\infty - \infty$, $0 \cdot \infty$, ∞/∞, and $0/0$. They are called *indeterminate forms*, and left undefined.

1.18 Principle of Mathematical Induction

The rigorous construction of the real number system starts with a set $\mathbb{N}$ of undefined elements called natural numbers, with the following properties. The set of natural number $\mathbb{N}$ satisfy the following:

- $\mathbb{N}$ is nonempty.
- Associated with each natural number n there is a unique natural number n' called the *successor of* n.
- There is a natural number $\overline{n}$ that is not the successor of any natural number.

The set of natural number $\mathbb{N}$ satisfy the following:

- Distinct natural numbers have distinct successors; that is, if $n \neq m$, then $n' \neq m'$.
- The only subset of $\mathbb{N}$ that contains $\overline{n}$ and the successors of all its elements is $\mathbb{N}$ itself.

Theorem: Let P_1, P_2,..., P_n, ... be propositions, one for each positive integer, such that

- P_1 is true;
- for each positive integer n, P_n implies P_{n+1}.

Then P_n is true for each positive integer n.

Proof: Let

$$\mathbb{M} = \{n \in \mathbb{N} \text{ and } P_n \text{ is true}\}.$$

From first axiom of Peano's, $1 \in \mathbb{M}$, and from second axiom, $n + 1 \in \mathbb{M}$ whenever $n \in \mathbb{M}$.

Therefore, $\mathbb{M} = \mathbb{N}$, by fourth axiom.

Example: Let P_n be the proposition that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution: Then P_1 is the proposition that $1 = 1$, which is certainly true.

If P_n is true, then adding $n+1$ to both sides of the equation yields

$$\begin{aligned}
(1 + 2 + \cdots + n) + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\
&= (n+1)\left(\frac{n}{2} + 1\right) \\
&= \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)}{2},
\end{aligned}$$

Example: For each nonnegative integer n, let x_n be a real number and suppose that

$$|x_{n+1} - x_n| \le r|x_n - x_{n-1}|, \quad n \ge 1,$$

where r is a fixed positive number.

Show that

$$|x_n - x_{n-1}| \le r^{n-1}|x_1 - x_0| \quad \text{if} \quad n \ge 1.$$

Solution: For $n = 1$, 2, and 3, we find that

$$\begin{aligned}
|x_2 - x_1| &\le r|x_1 - x_0|, \\
|x_3 - x_2| &\le r|x_2 - x_1| \le r^2|x_1 - x_0|, \\
|x_4 - x_3| &\le r|x_3 - x_2| \le r^3|x_1 - x_0|.
\end{aligned}$$

- It is important to verify P_1, since P_n may imply P_{n+1} even if some or all of the propositions $P_1, P_2, \ldots, P_n, \ldots$ are false.

- Let P_n be the proposition that $2n - 1$ is divisible by 2. If P_n is true then P_{n+1} is also, since

  $$2n + 1 = (2n - 1) + 2.$$

  However, we cannot conclude that P_n is true for $n \ge 1$. In fact, P_n is false for every n.

1.18.1 Principle of Mathematical Induction For $\mathbb{Z}$

Theorem: Let n_0 be any integer (positive, negative, or zero).

Let P_{n_0}, P_{n_0+1}, $\ldots$, P_n, $\ldots$ be propositions, one for each integer $n \geq n_0$, such that

- P_{n_0} is true;
- for each integer $n \geq n_0$, P_n implies P_{n+1}.

Then P_n is true for every integer $n \geq n_0$.

Proof: For $m \geq 1$, let Q_m be the proposition defined by $Q_m = P_{m+n_0-1}$. Then $Q_1 = P_{n_0}$ is true by first part.

If $m \geq 1$ and $Q_m = P_{m+n_0-1}$ is true, then $Q_{m+1} = P_{m+n_0}$ is true by second part with n replaced by $m + n_0 - 1$.

Therefore, Q_m is true for all $m \geq 1$ by Mathematical induction Theorem with P replaced by Q and n replaced by m. This is equivalent to the statement that P_n is true for all $n \geq n_0$.

Example: Prove the proposition P_n given by

$$3n + 16 > 0, \qquad n \geq -5.$$

Example: Let P_n be the proposition that

$$n! - 3^n > 0, \qquad n \geq 7.$$

If P_n is true, then

$$\begin{aligned}
(n+1)! - 3^{n+1} &= n!(n+1) - 3^{n+1} \\
&> 3^n(n+1) - 3^{n+1} \quad \text{(by the induction assumption)} \\
&= 3^n(n-2).
\end{aligned}$$

Therefore, P_n implies P_{n+1} if $n > 2$. By trial and error, $n_0 = 7$ is the smallest integer such that P_{n_0} is true; hence, P_n is true for $n \geq 7$, by Theorem **??**.

Theorem: Let n_0 be any integer (positive, negative, or zero). Let P_{n_0}, P_{n_0+1},$\ldots$, P_n, $\ldots$ be propositions, one for each integer $n \geq n_0$, such that

- P_{n_0} is true;
- for $n \geq n_0$, P_{n+1} is true if P_{n_0}, P_{n_0+1},$\ldots$, P_n are all true.

Then P_n is true for $n \geq n_0$.

Example: Let $S = \{x \in \mathbb{R} \ : \ 0 < x < 1\}$, $T = \{x \in (0,1) : \ x \text{ is rational}\}$, and $U = \{x \in (0,1) : x \text{ is irrational}\}$. Then $S \supset T$ and $S \supset U$, and the inclusion is strict in both cases. The unions of pairs of these sets are

$$S \cup T = S, \quad S \cup U = S, \quad \text{and} \quad T \cup U = S,$$

and their intersections are

$$S \cap T = T, \quad S \cap U = U, \quad \text{and} \quad T \cap U = \emptyset.$$

1.19 Generalization of Union and Intersection

: If $\mathcal{F}$ is an arbitrary collection of sets, then $\cup\{S : S \in \mathcal{F}\}$ is the set of all elements that are members of at least one of the sets in $\mathcal{F}$, and $\cap\{S : S \in \mathcal{F}\}$ is the set of all elements that are members of every set in $\mathcal{F}$.

The union and intersection of finitely many sets $S_1, \ldots, S_n$ are also written as $\bigcup_{k=1}^{n} S_k$ and $\bigcap_{k=1}^{n} S_k$.

The union and intersection of an infinite sequence $\{S_k\}_{k=1}^{\infty}$ of sets are written as $\bigcup_{k=1}^{\infty} S_k$ and $\bigcap_{k=1}^{\infty} S_k$.

Example: If $\mathcal{F}$ is the collection of sets

$$S_\rho = \{x : \rho < x \leq 1 + \rho\},$$

where $0 < \rho \leq 1/2$, then

$$\bigcup\{S_\rho : S_\rho \in \mathcal{F}\} = \{x : 0 < x \leq 3/2\}$$

$$\bigcap\{S_\rho : S_\rho \in \mathcal{F}\} = \{x : 1/2 < x \leq 1\}.$$

Example: If, for each positive integer k, the set S_k is the set of real numbers that can be written as $x = m/k$ for some integer m,

then $\bigcup_{k=1}^{\infty} S_k$ is the set of rational numbers and $\bigcap_{k=1}^{\infty} S_k$ is the set of integers.

If a and b are in the extended reals and $a < b$, then the *open interval* (a, b) is defined by

$$(a, b) = \{x : a < x < b\}.$$

The open intervals (a, ∞) and $(-\infty, b)$ are *semi-infinite* if a and b are finite, and $(-\infty, \infty)$ is the entire real line.

ε-neighborhood: If x_0 is a real number and $\varepsilon > 0$, then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an *ε-neighborhood* of x_0. If a set S contains an ε-neighborhood of x_0, then S is a *neighborhood* of x_0, and x_0 is an *interior point* of S.

The set of interior points of S is the *interior* of S, denoted by S^0. If every point of S is an interior point (that is, $S^0 = S$), then S is *open*.

A set S is *closed* if S^c is open.

Example: An open interval (a, b) is an open set, because if $x_0 \in (a, b)$ and $\varepsilon \leq \min\{x_0 - a, b - x_0\}$, then

$$(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a, b).$$

The entire line $\mathbb{R} = (-\infty, \infty)$ is open, and therefore $\emptyset = \mathbb{R}^c$ is closed. However, $\emptyset$ is also open. for to deny this is to say that $\emptyset$ contains a point that is not an interior point, which is absurd because $\emptyset$ contains no points. Since $\emptyset$ is open, $\mathbb{R}(= \emptyset^c)$ is closed. Thus, $\mathbb{R}$ and $\emptyset$ are both open and closed. They are the only subsets of $\mathbb{R}$ with this property

A *deleted neighborhood* of a point x_0 is a set that contains every point of some neighborhood of x_0 except for x_0 itself.

For example,

$$S = \{x : 0 < |x - x_0| < \varepsilon\}$$

is a deleted neighborhood of x_0. We also say that it is a *deleted ε-neighborhood* of x_0.

Theorem: The following statements are true for arbitrary collections, finite or infinite, of open and closed sets.

- The union of open sets is open.
- The intersection of closed sets is closed.

Proof: Let $\mathcal{G}$ be a collection of open sets and

$$S = \cup\{G : G \in \mathcal{G}\}.$$

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in $\mathcal{G}$, and since G_0 is open, it contains some ε-neighborhood of x_0. Since $G_0 \subset S$, this ε-neighborhood is in S, which is consequently a neighborhood of x_0. Thus, S is a neighborhood of each of its points, and therefore open, by definition.

Let $\mathcal{F}$ be a collection of closed sets and $T = \cap\{F : F \in \mathcal{F}\}$.

Then $T^c = \cup F^c : F \in \mathcal{F}$ and, since each F^c is open, T^c is open, from the first part. Therefore, T is closed, by definition.

Examples: If $-\infty < a < b < \infty$, the set

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

is closed, since its complement is the union of the open sets $(-\infty, a)$ and (b, ∞). We say that $[a, b]$ is a *closed interval*.

Examples: The set

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

is a *half-closed* or *half-open interval*

If $-\infty < a < b < \infty$, as is

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\};$$

however, neither of these sets is open or closed. (Why not?)

Examples: *Semi-infinite closed intervals* are sets of the form

$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$

$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\},$$

where a is finite. They are closed sets, since their complements are the open intervals $(-\infty, a)$ and (a, ∞), respectively.

Let S be a subset of $\mathbb{R}$. Then

- x_0 is a *limit point* of S if every deleted neighborhood of x_0 contains a point of S.
- x_0 is a *boundary point* of S if every neighborhood of x_0 contains at least one point in S and one not in S. The set of boundary points of S is the *boundary* of S, denoted by ∂S. The *closure* of S, denoted by $\overline{S}$, is $\overline{S} = S \cup \partial S$.

Let S be a subset of $\mathbb{R}$. Then

- x_0 is an *isolated point* of S if $x_0 \in S$ and there is a neighborhood of x_0 that contains no other point of S.

Let S be a subset of $\mathbb{R}$. Then

- x_0 is *exterior* to S if x_0 is in the interior of S^c. The collection of such points is the *exterior* of S.

Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Then

- The set of limit points of S is $(-\infty, -1] \cup [1, 2]$.
- $\partial S = \{-1, 1, 2, 3\}$ and $\overline{S} = (-\infty, -1] \cup [1, 2] \cup \{3\}$.

Let $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Then

- 3 is the only isolated point of S.

- The exterior of S is $(-1, 1) \cup (2, 3) \cup (3, \infty)$.

For $n \geq 1$, let

$$I_n = \left[\frac{1}{2n+1}, \frac{1}{2n}\right] \quad \text{and} \quad S = \bigcup_{n=1}^{\infty} I_n.$$

Then

- The set of limit points of S is $S \cup \{0\}$.
- $\partial S = x : x = 0$ or $x = 1/n$ $(n \geq 2)$ and $\overline{S} = S \cup \{0\}$.
- S has no isolated points.
- The exterior of S is

$$(-\infty, 0) \cup \left[\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right)\right] \cup \left(\frac{1}{2}, \infty\right).$$

1.20 Open Coverings

A collection $\mathcal{H}$ of open sets is an *open covering* of a set S if every point in S is contained in a set H belonging to $\mathcal{H}$; that is, if

$$S \subset \cup\{H \in \mathcal{H} : H \in \mathcal{H}\}.$$

Example: The set

$$S_1 = [0, 1],$$

is covered by the family of open intervals

$$\begin{aligned} \mathcal{H}_1 &= \left\{\left(x - \frac{1}{N}, x + \frac{1}{N}\right) \middle| 0 < x < 1\right\}, \\ &\quad (N = \text{positive integer}), \quad . \end{aligned}$$

Example: The set

$$S_2 = \{1, 2, \ldots, n, \ldots\},$$

is covered by the family of open intervals

$$\mathcal{H}_2 = \left\{\left(n - \frac{1}{4}, n + \frac{1}{4}\right) \middle| n = 1, 2, \ldots\right\}.$$

Example: The sets

$$S_3 = \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}, \quad \text{and} \quad S_4 = (0, 1)$$

are covered by the families of open intervals

$$\mathcal{H}_3 = \left\{\left(\frac{1}{n+\frac{1}{2}}, \frac{1}{n-\frac{1}{2}}\right) \middle| n = 1, 2, \ldots\right\},$$

and

$$\mathcal{H}_4 \quad = \quad \{(0,\rho)|\, 0 < \rho < 1\},$$

respectively.

Theorem: If $\mathcal{H}$ is an open covering of a closed and bounded subset S of the real numbers then S has an open covering $\widetilde{\mathcal{H}}$ consisting of finitely many open sets belonging to $\mathcal{H}$.

Compact set: A closed and bounded set of real numbers is called *compact*.

The Heine-Borel theorem says that any open covering of a compact set S contains a finite collection that also covers S.

This theorem and its converse show that we could just as well define a set S of reals to be compact if it has the Heine-Borel property; that is, if every open covering of S contains a finite subcovering.

The same is true of $\mathbb{R}^n$. This definition generalizes to more abstract spaces (called *topological spaces*) for which the concept of boundedness need not be defined.

Example: The set

$$S_1 = [0,1],$$

is covered by the family of open intervals

$$\begin{aligned}\mathcal{H}_1 \quad &= \quad \left\{\left(x - \frac{1}{N}, x + \frac{1}{N}\right) \middle|\, 0 < x < 1\right\}, \\ &\qquad (N = \text{positive integer}), \quad .\end{aligned}$$

The $2N$ intervals from $\mathcal{H}_1$ centered at the points $x_k = k/2N\,(0 \le k \le 2N-1)$ cover S_1.

Example: The set

$$S_2 = \{1, 2, \ldots, n, \ldots\},$$

$$S_3 = \left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}, \quad \text{and} \quad S_4 = (0,1)$$

are not compact sets.

Theorem: A set S is closed if and only if no point of S^c is a limit point of S.

Corollary: A set S is closed if and only if S contains all of its limit points.

Remark: Hence, a set with no limit points is closed according to the corollary, in agreement with Theorem. For example, any finite set is closed. More generally, S is closed if there is a $\delta > 0$ such $|x - y| \ge \delta$ for every pair $\{x, y\}$ of distinct points in S.

Chapter 2

Sequences and Series

2.1 Sequences

An *infinite sequence* (more briefly, a *sequence*) of real numbers is a real-valued function defined on a set of integers $n : n \geq k$. We call the values of the function the *terms* of the sequence.

We denote a sequence by listing its terms in order; thus,

$$\{s_n\}_k^\infty = \{s_k, s_{k+1}, \dots\}.$$

Examples: Consider the following sequences

$$\begin{aligned}
\left\{\frac{1}{n^2+1}\right\}_0^\infty &= \left\{1, \frac{1}{2}, \frac{1}{5}, \dots, \frac{1}{n^2+1}, \dots\right\}, \\
\{(-1)^n\}_0^\infty &= \{1, -1, 1, \dots, (-1)^n, \dots\}, \\
\left\{\frac{1}{n-2}\right\}_3^\infty &= \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-2}, \dots\right\}.
\end{aligned}$$

The real number s_n is the nth *term* of the sequence.
Examples: Usually we are interested only in the terms of a sequence and the order in which they appear, but not in the particular value of k in (**??**). Therefore, we regard the sequences The sequences are

$$\left\{\frac{1}{n-2}\right\}_3^\infty \quad \text{and} \quad \left\{\frac{1}{n}\right\}_1^\infty$$

identical.

In the absence of any indication to the contrary, we take $k = 0$ unless s_n is given by a rule that is invalid for some nonnegative integer, in which case k is understood to be the smallest positive integer such that s_n is defined for all $n \geq k$. For example, if

$$s_n = \frac{1}{(n-1)(n-5)},$$

then $k = 6$.

2.1.1 Convergent Sequence

A sequence $\{s_n\}$ *converges to a limit* L if for every $\varepsilon > 0$ there is an integer N such that

$$|s_n - L| < \varepsilon \quad \text{if} \quad n \geq N.$$

In this case we say that $\{s_n\}$ is *convergent* and write

$$\lim_{n\to\infty} s_n = L.$$

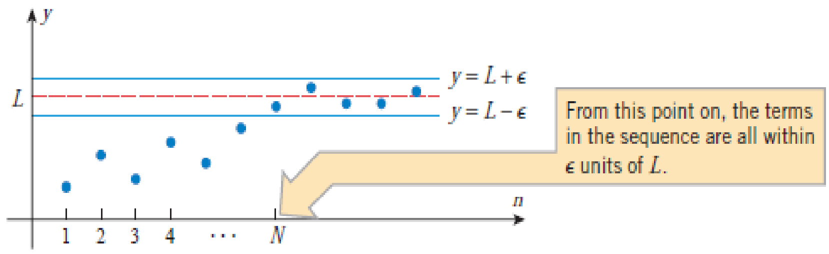


Figure 2.1: Convergence of the sequence

A sequence that does not converge *diverges*, or is *divergent*

Example: If $s_n = c$ for $n \geq k$, then $|s_n - c| = 0$ for $n \geq k$, and $\lim_{n\to\infty} s_n = c$.

Example: If

$$s_n = \left\{\frac{2n+1}{n+1}\right\},$$

then $\lim_{n\to\infty} s_n = 2$, since

$$\begin{aligned} |s_n - 2| &= \left|\frac{2n+1}{n+1} - \frac{2n+2}{n+1}\right| \\ &= \frac{1}{n+1}; \end{aligned}$$

Theorem: The limit of a convergent sequence is unique.
Proof: Suppose that

$$\lim_{n\to\infty} s_n = s \quad \text{and} \quad \lim_{n\to\infty} s_n = s'.$$

We must show that $s = s'$. Let $\varepsilon > 0$. From definition, there are integers N_1 and N_2 such that

$$|s_n - s| < \varepsilon \quad \text{if} \quad n \geq N_1$$

(because $\lim_{n\to\infty} s_n = s$), and

$$|s_n - s'| < \varepsilon \quad \text{if} \quad n \geq N_2$$

(because $\lim_{n\to\infty} s_n = s'$). These inequalities both hold if $n \geq N = \max(N_1, N_2)$, which implies that

$$\begin{aligned} |s - s'| &= |(s - s_N) + (s_N - s')| \\ &\leq |s - s_N| + |s_N - s'| < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Since this inequality holds for every $\varepsilon > 0$ and $|s - s'|$ is independent of ε, we conclude that $|s - s'| = 0$; that is, $s = s'$.

A sequence $\{s_n\}$ is said to be divergent to ∞ if for any real number a, $s_n > a$ for large n and written as

$$\lim_{n\to\infty} s_n = \infty$$

Similarly,

$$\lim_{n\to\infty} s_n = -\infty$$

if for any real number a, $s_n < a$ for large n. not regard $\{s_n\}$ as convergent unless $\lim_{n\to\infty} s_n$ is finite, as required by Definition of limit. To emphasize this distinction, we say that $\{s_n\}$ *diverges to* ∞ $(-\infty)$ if $\lim_{n\to\infty} s_n = \infty$ $(-\infty)$.

Example: The sequence $\{n/2 + 1/n\}$ diverges to ∞, since, if a is any real number, then

$$\frac{n}{2} + \frac{1}{n} > a \quad \text{if} \quad n \geq 2a.$$

Therefore, we write

$$\lim_{n\to\infty} \left(\frac{n}{2} + \frac{1}{n}\right) = \infty$$

Example: The sequence $\{n - n^2\}$ diverges to $-\infty$, since, if a is any real number, then

$$-n^2 + n = -n(n-1) < a \quad \text{if} \quad n > 1 + \sqrt{|a|}.$$

Therefore, we write

$$\lim_{n\to\infty} (-n^2 + n) = -\infty.$$

Example: The sequence $\{(-1)^n n^3\}$ diverges, but not to $-\infty$ or ∞.

2.2 Bounded Sequence

A sequence $\{s_n\}$ is *bounded above* if there is a real number b such that

$$s_n \leq b \quad \text{for all } n,$$

bounded below if there is a real number a such that

$$s_n \geq a \quad \text{for all } n,$$

or *bounded* if there is a real number r such that

$$|s_n| \leq r \quad \text{for all } n.$$

Example: If $s_n = [1+(-1)^n]n$, then $\{s_n\}$ is bounded below ($s_n \geq 0$) but unbounded above.

$\{-s_n\}$ is bounded above ($-s_n \leq 0$) but unbounded below.

If $s_n = (-1)^n$, then $\{s_n\}$ is bounded.

If $s_n = (-1)^n n$, then $\{s_n\}$ is not bounded above or below.

Theorem: A convergent sequence is bounded.
Proof: By taking $\varepsilon = 1$, we see that if $\lim_{n\to\infty} s_n = s$, then there is an integer N such that

$$|s_n - s| < 1 \quad \text{if} \quad n \geq N.$$

Therefore,

$$|s_n| = |(s_n - s) + s| \leq |s_n - s| + |s| < 1 + |s| \quad \text{if} \quad n \geq N,$$

and

$$|s_n| \leq \max\{|s_0|, |s_1|, \ldots, |s_{N-1}|, 1 + |s|\}$$

for all n, so $\{s_n\}$ is bounded.

Theorem: A sequence $\{s_n\}$ converges to s if and only if every neighborhood of s contains s_n for all but finitely many n.

2.3 Monotonic Sequences

A sequence $\{s_n\}$ is *nondecreasing* if $s_n \geq s_{n-1}$ for all n, or *nonincreasing* if $s_n \leq s_{n-1}$ for all n.

A *monotonic sequence* is a sequence that is either nonincreasing or nondecreasing. If $s_n > s_{n-1}$ for all n, then $\{s_n\}$ is *increasing*, while if $s_n < s_{n-1}$ for all n, $\{s_n\}$ is *decreasing*.

Theorem:

- If $\{s_n\}$ is nondecreasing, then $\lim_{n\to\infty} s_n = \sup\{s_n\}$.
- If $\{s_n\}$ is nonincreasing, then $\lim_{n\to\infty} s_n = \inf\{s_n\}$.

Recall: The definition of supremum and infimum of a set.
Proof: Let $\beta = \sup\{s_n\}$. If $\beta < \infty$, then by definition if $\varepsilon > 0$ then

$$\beta - \varepsilon < s_N \leq \beta$$

for some integer N.

Since $s_N \leq s_n \leq \beta$ if $n \geq N$, it follows that

$$\beta - \varepsilon < s_n \leq \beta \quad \text{if} \quad n \geq N.$$

This implies that $|s_n - \beta| < \varepsilon$ if $n \geq N$, so $\lim_{n\to\infty} s_n = \beta$, by definition.

If $\beta = \infty$ and b is any real number, then $s_N > b$ for some integer N. Then $s_n > b$ for $n \geq N$, so $\lim_{n\to\infty} s_n = \infty$.

For the proof of the second part try yourself.

Example: If $s_0 = 1$ and $s_n = 1 - e^{-s_{n-1}}$, then $0 < s_n \leq 1$ for all n, by induction.

$$\text{Since} \quad s_{n+1} - s_n = -(e^{-s_n} - e^{-s_{n-1}}) \quad \text{if} \quad n \geq 1.$$

The mean value theorem implies that

$$s_{n+1} - s_n = e^{-t_n}(s_n - s_{n-1}) \quad \text{if} \quad n \geq 1, \tag{2.1}$$

where t_n is between s_{n-1} and s_n. Since $s_1 - s_0 = -1/e < 0$, it follows by induction from (2.1) that $s_{n+1} - s_n < 0$ for all n.

Hence, $\{s_n\}$ is bounded and decreasing, and therefore convergent.

Remark: Let $\{x_n\}$ and $\{s_n\}$ be two sequences. If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number, and if

$$s_n \to 0,$$

then

$$x_n \to 0.$$

2.4 Some Special Sequences

Theorem: If $p > 0$, then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0.$$

Proof: Before presenting the proof let us recall the following:

Recall the binomial theorem:

$$(1+x)^n = 1 + nx + x(n-1)\frac{x^2}{2!} + ...$$

We have the following inequality

$$1 + nx \leq (1+x)^n, x > 0.$$

Theorem: If $p > 0$, then

$$\lim_{n\to\infty} \sqrt[n]{p} = 1.$$

Proof: We will discuss the three cases when $p = 1$, when $p > 1$ and when $0 < p < 1$. If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and, by the binomial theorem

$$1 + nx_n \leq (1+x_n)^n = p,$$

so that

$$0 < x_n \leq \frac{p-1}{n}.$$

Theorem: Show that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Proof: Take $x_n = \sqrt[n]{n} - 1$.

Then $x_n \geq 0$, and, by the binomial theorem

$$n = (1+x_n)^n \geq \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}, \qquad n \geq 2.$$

Theorem: If $p > 0$ and α is real, then

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0.$$

Proof: Let k be an integer such that $k > \alpha$, $k > 0$.

For $n > 2k$,

$$(1+p)^n > \frac{n(n-1)...(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k}, \quad n > 2k.$$

Since $\alpha - k < 0, n^{\alpha-k} \to 0$ by

$$\lim_{n\to\infty} \frac{1}{n^p} = 0.$$

Theorem: If $|x| < 1$, then

$$\lim_{n\to\infty} x^n = 0.$$

Theorem: Let $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, where s and t are finite. Then

$$\lim_{n\to\infty} (cs_n) = cs \tag{2.2}$$

if c is a constant;

$$\begin{aligned}
\lim_{n\to\infty} (s_n + t_n) &= s + t, \\
\lim_{n\to\infty} (s_n - t_n) &= s - t, \\
\lim_{n\to\infty} (s_n t_n) &= st,
\end{aligned}$$

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{s}{t}$$

if t_n is nonzero for all n and $t \neq 0$.

Proof: We write

$$s_n t_n - st = s_n t_n - st_n + st_n - st = (s_n - s)t_n + s(t_n - t);$$

hence,

$$|s_n t_n - st| \leq |s_n - s|\,|t_n| + |s|\,|t_n - t|.$$

Since $\{t_n\}$ converges, it is bounded.

Therefore, there is a number R such that $|t_n| \leq R$ for all n, and (**??**) implies that

$$|s_n t_n - st| \leq R|s_n - s| + |s|\,|t_n - t|. \tag{2.3}$$

By definition, if $\varepsilon > 0$ there are integers N_1 and N_2 such that

$$\begin{aligned} |s_n - s| &< \varepsilon \quad \text{if} \quad n \geq N_1 \\ |t_n - t| &< \varepsilon \quad \text{if} \quad n \geq N_2. \end{aligned}$$

If $N = \max(N_1, N_2)$, then both inequalities hold when $n \geq N$, and the (2.3) implies that

$$|s_n t_n - st| \leq (R + |s|)\varepsilon \quad \text{if} \quad n \geq N.$$

This proves (2.3).

We consider the special case where $s_n = 1$ for all n and $t \neq 0$; thus, we want to show that

$$\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{t}.$$

First, observe that since $\lim_{n\to\infty} t_n = t \neq 0$, there is an integer M such that $|t_n| \geq |t|/2$ if $n \geq M$.

By definition with $\varepsilon = |t|/2$; thus, there is an integer M such that $|t_n - t| < |t/2|$ if $n \geq M$.

Therefore,

$$|t_n| = |t + (t_n - t)| \geq ||t| - |t_n - t|| \geq \frac{|t|}{2} \quad \text{if} \quad n \geq M.$$

If $\varepsilon > 0$, choose N_0 so that $|t_n - t| < \varepsilon$ if $n \geq N_0$, and let $N = \max(N_0, M)$. Then

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \frac{|t - t_n|}{|t_n|\,|t|} \leq \frac{2\varepsilon}{|t|^2} \quad \text{if} \quad n \geq N.$$

hence, $\lim_{n\to\infty} 1/t_n = 1/t$. Now we obtain (2.3) in the general case from (2.3) with $\{t_n\}$ replaced by $\{1/t_n\}$.

Example: Find the limit of the sequence

$$s_n = \frac{1}{n}\sin\frac{n\pi}{4} + \frac{2(1 + 3/n)}{1 + 1/n}.$$

Solution: We apply the applicable parts of Theorem as follows:

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{1}{n}\sin\frac{n\pi}{4} + \frac{2\left[\lim_{n\to\infty} 1 + 3\lim_{n\to\infty}(1/n)\right]}{\lim_{n\to\infty}} 1 + \lim_{n\to\infty}(1/n) \\ &= 0 + \frac{2(1+3\cdot 0)}{1+0} = 2.\end{aligned}$$

Example: Find the limit of the sequence $s_n = \lim_{n\to\infty}\frac{(n/2)+\log n}{3n+4\sqrt{n}}$.
Solution:

$$\begin{aligned}\lim_{n\to\infty}\frac{(n/2)+\log n}{3n+4\sqrt{n}} &= \lim_{n\to\infty}\frac{1/2+(\log n)/n}{3+4n^{-1/2}} \\ &= \frac{\lim_{n\to\infty} 1/2 + \lim_{n\to\infty}(\log n)/n}{\lim_{n\to\infty} 3 + 4\lim_{n\to\infty} n^{-1/2}} \\ &= \frac{1/2+0}{3+0} \\ &= \frac{1}{6}.\end{aligned}$$

2.5 Subsequence

A sequence $\{t_k\}$ is a *subsequence* of a sequence $\{s_n\}$ if

$$t_k = s_{n_k}, \quad k \geq 0,$$

where $\{n_k\}$ is an increasing infinite sequence of integers in the domain of $\{s_n\}$. We denote the subsequence $\{t_k\}$ by $\{s_{n_k}\}$.
Example: If

$$\{s_n\} = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\},$$

then letting $n_k = 2k$ yields the subsequence

$$\{s_{2k}\} = \left\{\frac{1}{2k}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2k}, \ldots\right\},$$

and letting $n_k = 2k+1$ yields the subsequence

$$\{s_{2k+1}\} = \left\{\frac{1}{2k+1}\right\} = \left\{1, \frac{1}{3}, \ldots, \frac{1}{2k+1}, \ldots\right\}.$$

Example: The sequence $\{s_n\}$ defined by

$$s_n = (-1)^n\left(1+\frac{1}{n}\right)$$

does not converge, but $\{s_n\}$ has subsequences that do. For example,

$$\{s_{2k}\} = \left\{1+\frac{1}{2k}\right\}$$

$$\lim_{k\to\infty} s_{2k} = 1.$$

Example: The sequence $\{s_n\}$ defined by

$$s_n = (-1)^n\left(1+\frac{1}{n}\right)$$

does not converge, but $\{s_n\}$ has subsequences that do.

For example,

$$\{s_{2k+1}\} = \left\{-1-\frac{1}{2k+1}\right\}$$

$$\lim_{k\to\infty} s_{2k+1} = -1.$$

It can be shown that a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ converges to 1 if and only if n_k is even for k sufficiently large, or to -1 if and only if n_k is odd for k sufficiently large. Otherwise, $\{s_{n_k}\}$ diverges.

Theorem: If

$$\lim_{n\to\infty} s_n = s \quad (-\infty \le s \le \infty),$$

then

$$\lim_{k\to\infty} s_{n_k} = s$$

for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$.

Proof: By definition for every $\varepsilon > 0$, there is an integer N such that

$$|s_n - s| < \varepsilon \quad \text{if} \quad n \ge N.$$

Since $\{n_k\}$ is an increasing sequence, there is an integer K such that $n_k \ge N$ if $k \ge K$. Therefore,

$$|s_{n_k} - L| < \varepsilon \quad \text{if} \quad k \ge K.$$

Theorem: If $\{s_n\}$ is monotonic and has a subsequence $\{s_{n_k}\}$ such that

$$\lim_{k\to\infty} s_{n_k} = s,$$

then

$$\lim_{n\to\infty} s_n = s.$$

Recall the following theorem:

If $\{s_n\}$ is nondecreasing, then $\lim_{n\to\infty} s_n = \sup\{s_n\}$.

If $\{s_n\}$ is nonincreasing, then $\lim_{n\to\infty} s_n = \inf\{s_n\}$.

Proof: Since $\{s_{n_k}\}$ is also nondecreasing in this case, it is sufficient to show that

$$\sup\{s_{n_k}\} = \sup\{s_n\}$$

Since the set of terms of $\{s_{n_k}\}$ is contained in the set of terms of $\{s_n\}$,

$$\sup\{s_n\} \geq \sup\{s_{n_k}\}.$$

Since $\{s_n\}$ is nondecreasing, there is for every n an integer n_k such that $s_n \leq s_{n_k}$. This implies that

$$\sup\{s_n\} \leq \sup\{s_{n_k}\}.$$

Theorem: A point $\overline{x}$ is a limit point of a set S **if and only if** there is a sequence $\{x_n\}$ of points in S such that $x_n \neq \overline{x}$ for $n \geq 1$, and

$$\lim_{n\to\infty} x_n = \overline{x}.$$

Proof: By definition for each $\varepsilon > 0$, there is an integer N such that $0 < |x_n - \overline{x}| < \varepsilon$ if $n \geq N$.

Therefore, every ε-neighborhood of $\overline{x}$ contains infinitely many points of S. This means that $\overline{x}$ is a limit point of S.

For necessity, let $\overline{x}$ be a limit point of S. Then, for every integer $n \geq 1$, the interval $(\overline{x} - 1/n, \overline{x} + 1/n)$ contains a point x_n $(\neq \overline{x})$ in S.

Since $|x_m - \overline{x}| \leq 1/n$ if $m \geq n$, $\lim_{n\to\infty} x_n = \overline{x}$.

Theorem:

- If $\{x_n\}$ is bounded, then $\{x_n\}$ has a convergent subsequence.
- If $\{x_n\}$ is unbounded above, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

- If $\{x_n\}$ is unbounded below, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} x_{n_k} = -\infty.$$

Proof: Let S be the set of distinct numbers that occur as terms of $\{x_n\}$.

(For example, if $\{x_n\} = \{(-1)^n\}$, $S = \{1, -1\}$; if $\{x_n\} = \{1, \frac{1}{2}, 1, \frac{1}{3}, \ldots, 1, 1/n, \ldots\}$, $S = \{1, \frac{1}{2}, \ldots, 1/n, \ldots\}$.)

If S contains only finitely many points, then some $\overline{x}$ in S occurs infinitely often in $\{x_n\}$; that is, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = \overline{x}$ for all k. Then $\lim_{k\to\infty} x_{n_k} = \overline{x}$, and we are finished in this case.

If S is infinite, then, since S is bounded (by assumption), the Bolzano-Weierstrass theorem implies that S has a limit point $\overline{x}$.

There is a sequence of points $\{y_j\}$ in S, distinct from $\overline{x}$, such that

$$\lim_{j\to\infty} y_j = \overline{x}.$$

Although each y_j occurs as a term of $\{x_n\}$, $\{y_j\}$ is not necessarily a subsequence of $\{x_n\}$, because if we write

$$y_j = x_{n_j}.$$

There is no reason to expect that $\{n_j\}$ is an increasing sequence as required in definition of subsequence.

However, it is always possible to pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ that is increasing, and then the sequence $\{y_{j_k}\} = \{s_{n_{j_k}}\}$ is a subsequence of both $\{y_j\}$ and $\{x_n\}$.

2.6 Limit Superior and Limit Inferior

Theorem:

- If $\{s_n\}$ is bounded above and does not diverge to $-\infty$, then there is a unique real number $\overline{s}$ such that, if $\varepsilon > 0$,

$$s_n < \overline{s} + \varepsilon \quad \text{for large } n \tag{2.4}$$

and

$$s_n > \overline{s} - \varepsilon \quad \text{for infinitely many } n. \tag{2.5}$$

- If $\{s_n\}$ is bounded below and does not diverge to ∞, then there is a unique real number $\underline{s}$ such that, if $\varepsilon > 0$,

$$s_n > \underline{s} - \varepsilon \quad \text{for large } n \tag{2.6}$$

and

$$s_n < \underline{s} + \varepsilon \quad \text{for infinitely many } n. \tag{2.7}$$

Proof: Proof of the first part. Since $\{s_n\}$ is bounded above, there is a number β such that $s_n < \beta$ for all n. Since $\{s_n\}$ does not diverge to $-\infty$, there is a number α such that $s_n > \alpha$ for infinitely many n. If we define

$$M_k = \sup\{s_k, s_{k+1}, \dots, s_{k+r}, \dots\},$$

then $\alpha \leq M_k \leq \beta$, so $\{M_k\}$ is bounded. Since $\{M_k\}$ is nonincreasing (why?), it converges. Let

$$\overline{s} = \lim_{k\to\infty} M_k. \tag{2.8}$$

If $\varepsilon > 0$, then $M_k < \overline{s} + \varepsilon$ for large k, and since $s_n \leq M_k$ for $n \geq k$, $\overline{s}$ satisfies (2.4).

If (2.5) were false for some positive ε, there would be an integer K such that

$$s_n \leq \overline{s} - \varepsilon \quad \text{if} \quad n \geq K.$$

However, this implies that

$$M_k \leq \overline{s} - \varepsilon \quad \text{if} \quad k \geq K,$$

which contradicts (2.8). Therefore, $\overline{s}$ has the stated properties.

Now we must show that $\overline{s}$ is the only real number with the stated properties. If $t < \overline{s}$, the inequality

$$s_n < t + \frac{\overline{s} - t}{2} = \overline{s} - \frac{\overline{s} - t}{2}$$

cannot hold for all large n, because this would contradict (2.5) with $\varepsilon = (\overline{s} - t)/2$. If $\overline{s} < t$, the inequality

$$s_n > t - \frac{t - \overline{s}}{2} = \overline{s} + \frac{t - \overline{s}}{2}$$

cannot hold for infinitely many n, because this would contradict (2.4) with $\varepsilon = (t - \overline{s})/2$. Therefore, $\overline{s}$ is the only real number with the stated properties.

The numbers $\overline{s}$ and $\underline{s}$ defined in the previous Theorem are called the *limit superior* and *limit inferior*, respectively, of $\{s_n\}$, and denoted by

$$\overline{s} = \limsup_{n\to\infty} s_n \quad \text{and} \quad \underline{s} = \liminf_{n\to\infty} s_n.$$

We also define

$$\begin{aligned}
\limsup_{n\to\infty} s_n &= \infty && \text{if } \{s_n\} \text{ is not bounded above,}\\
\limsup_{n\to\infty} s_n &= -\infty && \text{if } \lim_{n\to\infty} s_n = -\infty,\\
\liminf_{n\to\infty} s_n &= -\infty && \text{if } \{s_n\} \text{ is not bounded below,}\\
\liminf_{n\to\infty} s_n &= \infty && \text{if } \lim_{n\to\infty} s_n = \infty.
\end{aligned}$$

Theorem: Every sequence $\{s_n\}$ of real numbers has a unique limit superior, $\overline{s}$, and a unique limit inferior, $\underline{s}$, in the extended reals, and

$$\underline{s} \le \overline{s}.$$

Examples:

$$\limsup_{n\to\infty} r^n = \begin{cases} \infty, & |r| > 1,\\ 1, & |r| = 1,\\ 0, & |r| < 1; \end{cases}$$

and

$$\liminf_{n\to\infty} r^n = \begin{cases} \infty, & r > 1,\\ 1, & r = 1,\\ 0, & |r| < 1,\\ -1, & r = -1,\\ -\infty, & r < -1. \end{cases}$$

Example:

$$\begin{aligned}
\limsup_{n\to\infty} n^2 = \liminf_{n\to\infty} n^2 &= \infty, \\
\limsup_{n\to\infty} (-1)^n \left(1 - \frac{1}{n}\right) &= 1, \\
\liminf_{n\to\infty} (-1)^n \left(n - \frac{1}{n}\right) &= -1,
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \left[1 + (-1)^n\right] n^2 &= \infty, \\
\liminf_{n\to\infty} \left[1 + (-1)^n\right] n^2 &= 0.
\end{aligned}$$

Theorem: If $\{s_n\}$ is a sequence of real numbers, then

$$\lim_{n\to\infty} s_n = s \tag{2.9}$$

if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s. \tag{2.10}$$

Proof: If $s = \pm\infty$, the equivalence of (2.9) and (2.10) follows immediately from their definitions. If $\lim_{n\to\infty} s_n = s$ (finite), then definition of a convergent sequence implies that (2.4)–(2.7) hold with $\overline{s}$ and $\underline{s}$ replaced by s. Hence, (2.10) follows from the uniqueness of $\overline{s}$ and $\underline{s}$. For the converse, suppose that $\overline{s} = \underline{s}$ and let s denote their common value. Then (2.4) and (2.6) imply that

$$s - \varepsilon < s_n < s + \varepsilon$$

for large n, and (2.9) follows from Definition and the uniqueness of $\lim_{n\to\infty} s_n$.

2.7 Cauchy Sequence

A sequence $\{s_n\}$ of real numbers is called a Cauchy sequence if for every $\varepsilon > 0$, there is an integer N such that

$$|s_n - s_m| < \varepsilon \quad \text{if} \quad m, n \geq N.$$

Theorem: If $\{s_n\}$ is a Cauchy sequence of real numbers, then $\{s_n\}$ is bounded.

Proof: See Lecture

Theorem (**Cauchy's convergence criterion**): A sequence $\{s_n\}$ of real numbers converges **if and only if**, for every $\varepsilon > 0$, there is an integer N such that

$$|s_n - s_m| < \varepsilon \quad \text{if} \quad m, n \geq N.$$

Proof: Suppose that $\lim_{n\to\infty} s_n = s$ and $\varepsilon > 0$.

There is an integer N such that

$$|s_r - s| < \frac{\varepsilon}{2} \quad \text{if} \quad r \geq N.$$

Therefore,

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m| < \varepsilon,$$

whenever $n, m \geq N$. Therefore, the stated condition is necessary for convergence of $\{s_n\}$.

Recall a Cauchy sequence $\{s_n\}$ is bounded. So $\overline{s}$ and $\underline{s}$ are finite.

Now suppose that $\varepsilon > 0$ and N satisfies $|s_n - s_m| < \varepsilon$, $n, m \geq N$.

From definition of limit superior and limit inferior

$$|s_n - \overline{s}| < \varepsilon, \qquad |s_m - \underline{s}| < \varepsilon$$

for some integer $m, n > N$.

Since

$$\begin{aligned} |\overline{s} - \underline{s}| &= |(\overline{s} - s_n) + (s_n - s_m) + (s_m - \underline{s})| \\ &\leq |\overline{s} - s_n| + |s_n - s_m| + |s_m - \underline{s}|. \end{aligned}$$

We have

$$|\overline{s} - \underline{s}| < 3\varepsilon.$$

Since ε is an arbitrary positive number, this implies that $\overline{s} = \underline{s}$, so $\{s_n\}$ converges.

2.8 Series

If $\{a_n\}_k^\infty$ is an infinite sequence of real numbers, the symbol

$$\sum_{n=k}^{\infty} a_n$$

is an *infinite series*, and a_n is the *nth term* of the series.

We say that $\sum_{n=k}^{\infty} a_n$ *converges to the sum* A, and write

$$\sum_{n=k}^{\infty} a_n = A,$$

if the sequence $\{A_n\}_k^\infty$ defined by

$$A_n = a_k + a_{k+1} + \cdots + a_n, \quad n \geq k,$$

converges to A.

2.8.1 Sequence of Partial Sums

The finite sum A_n is the *nth partial sum of* $\sum_{n=k}^{\infty} a_n$.

If $\{A_n\}_k^{\infty}$ diverges, we say that $\sum_{n=k}^{\infty} a_n$ *diverges*.

In particular, if $\lim_{n\to\infty} A_n = \infty$ or $-\infty$, we say that $\sum_{n=k}^{\infty} a_n$ *diverges to* ∞ *or* $-\infty$, and write

$$\sum_{n=k}^{\infty} a_n = \infty \quad \text{or} \quad \sum_{n=k}^{\infty} a_n = -\infty.$$

2.8.2 Oscillatory Series

A divergent infinite series that does not diverge to $\pm\infty$ is said to *oscillate*, or *be oscillatory*.

Example: Consider the series

$$\sum_{n=0}^{\infty} r^n, \quad -1 < r < 1.$$

Here $a_n = r^n$ $(n \geq 0)$. The nth term of sequence of partial sum is

$$A_n = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

which converges to $1/(1-r)$ as $n \to \infty$. Thus, we write

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad -1 < r < 1.$$

Example: The nth term of sequence of partial sum is

$$A_n = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

If $|r| > 1$, then $\sum_{n=0}^{\infty} r^n$ diverges; if $r > 1$, then

$$\sum_{n=0}^{\infty} r^n = \infty,$$

Example: The nth term of sequence of partial sum is

$$A_n = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r},$$

If $r < -1$, $\sum_{n=0}^{\infty} r^n$ oscillates, since its partial sums alternate in sign and their magnitudes become arbitrarily large for large n.

If $r = -1$, then $A_{2m+1} = 0$ and $A_{2m} = 1$ for $m \geq 0$, while if $r = 1$, $A_n = n + 1$; in both cases the series diverges.

Theorem: The sum of a convergent series is unique.
Proof: See Lecture.

Theorem: Let $\sum_{n=k}^{\infty} a_n = A$ and $\sum_{n=k}^{\infty} b_n = B$, where A and B are finite. Then

$$\sum_{n=k}^{\infty}(ca_n) = cA$$

if c is a constant,

$$\sum_{n=k}^{\infty}(a_n + b_n) = A + B,$$

and

$$\sum_{n=k}^{\infty}(a_n - b_n) = A - B.$$

These relations also hold if one or both of A and B is infinite, provided that the right sides are not indeterminate.

2.9 Dropping Finitely Many Terms

For example, suppose that we drop the first k terms of a series $\sum_{n=0}^{\infty} a_n$, and consider the new series $\sum_{n=k}^{\infty} a_n$. sums of the two series by

$$\begin{aligned} A_n &= a_0 + a_1 + \cdots + a_n, \quad n \geq 0, \\ A'_n &= a_k + a_{k+1} + \cdots + a_n, \quad n \geq k. \end{aligned}$$

$$A_n = (a_0 + a_1 + \cdots + a_{k-1}) + A'_n, \quad n \geq k,$$

it follows that $A = \lim_{n\to\infty} A_n$ exists (in the extended reals) if and only if $A' = \lim_{n\to\infty} A'_n$ does, and in this case

$$A = (a_0 + a_1 + \cdots + a_{k-1}) + A'.$$

An important principle follows from this.

Lemma: Suppose that for n sufficiently large (that is, for $n \geq N$) the terms of $\sum_{n=k}^{\infty} a_n$ satisfy some condition that implies convergence of an infinite series. Then $\sum_{n=k}^{\infty} a_n$ converges.

Similarly, suppose that for n sufficiently large the terms $\sum_{n=k}^{\infty} a_n$ satisfy some condition that implies divergence of an infinite series. Then $\sum_{n=k}^{\infty} a_n$ diverges.

Example: *The series* $\sum_{k}^{\infty} a_n$ *converges if* $(-1)^n a_n > 0$, $|a_{n+1}| < |a_n|$, *and* $\lim_{n\to\infty} a_n = 0$.

The terms of

$$\sum_{n=1}^{\infty} \frac{16+(-2)^n}{n2^n}$$

do not satisfy these conditions for all $n \geq 1$, but they do satisfy them for sufficiently large n.

Theorem: A series $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$|a_n + a_{n+1} + \cdots + a_m| < \varepsilon \quad \text{if} \quad m \geq n \geq N.$$

Proof: In terms of the partial sums $\{A_n\}$ of $\sum a_n$,

$$a_n + a_{n+1} + \cdots + a_m = A_m - A_{n-1}.$$

Therefore, (**??**) can be written as

$$|A_m - A_{n-1}| < \varepsilon \quad \text{if} \quad m \geq n \geq N.$$

Since $\sum a_n$ converges if and only if $\{A_n\}$ converges, Theorem implies the conclusion.

Theorem: Show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof: Consider the sequence of partial sums

$$s_1 = 1, s_2 = 1 + \frac{1}{2}, s_3 = 1 + \frac{1}{2} + \frac{1}{3}, s_4 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots$$

form strictly increasing sequence

$$s_1 < s_2 < s_3 < \ldots < s_n < \ldots$$

$$\begin{aligned}
s_{2^1} &= 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} \\
s_{4=2^2} &= s_2 + \frac{1}{3} + \frac{1}{4} > s_2 + \frac{1}{4} + \frac{1}{4} > \frac{4}{2} \\
s_{8=3^3} &= s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > s_4 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{4}{2} \\
\ldots s_{2^n} &> \frac{n+1}{2}.
\end{aligned}$$

If M is any constant, we can find a positive integer n such that

$$\frac{n+1}{2} > M.$$

But for this n, we have

$$s_{2^n} > \frac{n+1}{2} > M.$$

so that no constant M is greater than or equal to every partial sum of the harmonic series.

Example: Consider the geometric series $\sum r^n$.

If $|r| < 1$ and $m \geq n$, then

$$\begin{aligned} |A_m - A_n| &= |r^{n+1} + r^{n+2} + \cdots + r^m| \\ &\leq |r|^{n+1}(1 + |r| + \cdots + |r|^{m-n-1}) \\ &= |r|^{n+1}\frac{1 - |r|^{m-n}}{1 - |r|} < \frac{|r|^{n+1}}{1 - |r|}. \end{aligned}$$

If $\varepsilon > 0$, choose N so that

$$\frac{|r|^{N+1}}{1 - |r|} < \varepsilon.$$

Then Cauchy's convergence criterion implies that

$$|A_m - A_n| < \varepsilon \quad \text{if} \quad m \geq n \geq N.$$

Now Theorem implies that $\sum r^n$ converges if $|r| < 1$, as in Example **??**.

Corollary: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Proof: If $\sum a_n$ converges, then for each $\varepsilon > 0$ there is an integer K such that

$$\left|\sum_{n=k}^{\infty} a_n\right| < \varepsilon \quad \text{if} \quad k \geq K;$$

that is,

$$\lim_{k\to\infty} \sum_{n=k}^{\infty} a_n = 0.$$

Be careful: $\lim_{n\to\infty} a_n = 0$ Necessary condition.

Not sufficient.

Example: For the harmonic series $\lim_{n\to\infty} \frac{1}{n} = 0$.

But we have proved that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Corollary: If $\sum a_n$ converges, then for each $\varepsilon > 0$ there is an integer K such that

$$\left|\sum_{n=k}^{\infty} a_n\right| < \varepsilon \quad \text{if} \quad k \geq K;$$

that is,

$$\lim_{k\to\infty}\sum_{n=k}^{\infty} a_n = 0.$$

Example: If $|r| < 1$, then

$$\left|\sum_{n=k}^{\infty} r^n\right| = \left|r^k\sum_{n=k}^{\infty} r^{n-k}\right| = \left|r^k\sum_{n=0}^{\infty} r^n\right| = \frac{|r|^k}{1-r}.$$

Therefore, if

$$\frac{|r|^K}{1-r} < \varepsilon,$$

then

$$\left|\sum_{n=k}^{\infty} r^n\right| < \varepsilon \quad \text{if} \quad k \geq K,$$

which implies that $\lim_{k\to\infty}\sum_{n=k}^{\infty} r^n = 0$.

2.10 Series of Nonnegative Terms

The series $\sum a_n$ is said to be series of nonnegative terms if $a_n \geq 0$ for $n \geq k$. The theory of series $\sum a_n$ with terms that are nonnegative for sufficiently large n is simpler than the general theory, since such

The series of nonnegative terms either converges to a finite limit or diverges to ∞.

Recall the following theorem:

- If $\{s_n\}$ is nondecreasing, then $\lim_{n\to\infty} s_n = \sup\{s_n\}$.
- If $\{s_n\}$ is nonincreasing, then $\lim_{n\to\infty} s_n = \inf\{s_n\}$.

Theorem: If $a_n \geq 0$ for $n \geq k$, then $\sum a_n$ converges if its partial sums are bounded, or diverges to ∞ if they are not. These are the only possibilities and, in either case,

$$\sum_{n=k}^{\infty} a_n = \sup\{A_n : n \geq k\},$$

where

$$A_n = a_k + a_{k+1} + \cdots + a_n, \quad n \geq k.$$

Proof: Since $A_n = A_{n-1} + a_n$ and $a_n \geq 0$ $(n \geq k)$, the sequence $\{A_n\}$ is nondecreasing, so the conclusion follows from Theorem and definition of convergence of a series.

Recall the theorem: If $a_n \geq 0$ for $n \geq k$, then $\sum a_n$ converges if its partial sums are bounded, or diverges to ∞ if they are not. These are the only possibilities and, in either case,

$$\sum_{n=k}^{\infty} a_n = \sup\{A_n : n \geq k\},$$

where

$$A_n = a_k + a_{k+1} + \cdots + a_n, \quad n \ge k.$$

2.11 The Comparison Test

Theorem: Suppose that

$$0 \le a_n \le b_n, \quad n \ge k.$$

Then

- $\sum a_n < \infty$ if $\sum b_n < \infty$.
- $\sum b_n = \infty$ if $\sum a_n = \infty$.

Proof: If

$$A_n = a_k + a_{k+1} + \cdots + a_n \quad \text{and} \quad B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \ge k,$$

then, we have

$$A_n \le B_n.$$

If $\sum b_n < \infty$, then $\{B_n\}$ is bounded above implies that $\{A_n\}$ is also; therefore, $\sum a_n < \infty$.

On the other hand, if $\sum a_n = \infty$, then $\{A_n\}$ is unbounded above implies that $\{B_n\}$ is also; therefore, $\sum b_n = \infty$.

Example: Discuss the convergence of the series $\sum \frac{r^n}{n}$.

Solution: Since

$$\frac{r^n}{n} < r^n, \quad n \ge 1,$$

and $\sum r^n < \infty$ if $0 < r < 1$, the series $\sum r^n/n$ converges if $0 < r < 1$, by the comparison test.

Comparing these two series is **inconclusive** if $r > 1$, since it does not help to know that the terms of $\sum r^n/n$ are smaller than those of the divergent series $\sum r^n$.

If $r < 0$, the comparison test does not apply, since the series then have infinitely many negative terms. Since

$$r^n < nr^n$$

and $\sum r^n = \infty$ if $r \ge 1$, the comparison test implies that $\sum nr^n = \infty$ if $r \ge 1$.

Comparing these two series is inconclusive if $0 < r < 1$, since it does not help to know that the terms of $\sum nr^n$ are larger than those of the convergent series $\sum r^n$.

Recall the following: If $a_n \geq 0$ for $n \geq k$, then $\sum a_n$ converges if its partial sums are bounded, or diverges to ∞ if they are not. These are the only possibilities and, in either case,

$$\sum_{n=k}^{\infty} a_n = \sup\{A_n : n \geq k\},$$

where

$$A_n = a_k + a_{k+1} + \cdots + a_n, \quad n \geq k.$$

Theorem: Suppose $a_1 \geq a_2 \geq ... \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum\limits_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof: Consider the sequence of partial sum

$$s_n = a_1 + a_2 + ... + a_n, \qquad t_k = a_1 + 2a_2 + ... + 2^k a_{2^k}.$$

For $n < 2^k$, we have

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + ... + (a_{2^k} + ... + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + ... + 2^k a_{2^k} = t_k \end{aligned}$$

so that

$$s_n \leq t_k, \qquad (*).$$

On the other hand, if $n > 2^k$, we have

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + ... + (a_{2^{k-1}+1} + ... + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + ... + 2^{k-1}a_{2^k} = \frac{1}{2}t_k \end{aligned}$$

so that

$$2s_n \geq t_k \qquad (**).$$

By (*) and (**) both sequences $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded.

Recall the following theorem:

Theorem: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem: Suppose $a_1 \geq a_2 \geq ... \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum\limits_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Theorem: The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof: If $p = 0$, then $\sum \frac{1}{n^p}$ is divergent. If $p < 0$, then once again $\sum \frac{1}{n^p}$ is divergent. (Why) If $p > 0$ then the series $\sum \frac{1}{n^p}$ and the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Notice that $2^{1-p} < 1$ if and only if $1-p<0$. Take $x = 2^{1-p}$, then we have $|x|<1$ and the series $\sum\limits_{k=0}^{\infty} 2^{(1-p)k}$ becomes

$$\sum_{k=0}^{\infty} x^k.$$

The monotonicity of the logarithmic function implies $\{\log n\}$ increases. Hence $\{1/n\log n\}$ decreases.

Theorem: If $p>1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},$$

converges, if $p \leq 1$, the series diverges.

Proof: The monotonicity of the logarithmic function implies $\{\log n\}$ increases. Hence $\{1/n\log n\}$ decreases.

The series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},$$

and

$$\sum_{k=1}^{\infty} \frac{1}{2^k(k\log 2)^p} = \frac{1}{(\log 2)^p}\sum_{k=1}^{\infty}\frac{1}{k^p}.$$

Remark: Consider the series

$$\sum_{n=3}^{\infty} \frac{1}{n\log n\log\log n},$$

diverges whereas

$$\sum_{n=3}^{\infty} \frac{1}{n\log n(\log\log n)^2},$$

converges.

Example: Show that the series

$$\sum^{\infty} \frac{1}{(n^2+n)^q}$$

converges if $q>1/2$.

Solution: By comparison with the convergent series $\sum 1/n^{2q}$, since

$$\frac{1}{(n^2+n)^q} < \frac{1}{n^{2q}}, \quad n \geq 1.$$

This comparison is inconclusive if $q \leq 1/2$, since then

$$\sum \frac{1}{n^{2q}} = \infty,$$

and it does not help to know that the terms of $\sum^\infty \frac{1}{(n^2+n)^q}$ are smaller than those of a divergent series. However, we can use the comparison test here, after a little trick.

Observe that

$$\sum_{n=k-1}^{\infty} \frac{1}{(n+1)^{2q}} = \sum_{n=k}^{\infty} \frac{1}{n^{2q}} = \infty, \quad q \leq 1/2,$$

and

$$\frac{1}{(n+1)^{2q}} < \frac{1}{(n^2+n)^q}.$$

Therefore, the comparison test implies that

$$\sum \frac{1}{(n^2+n)^q} = \infty, \quad q \leq 1/2.$$

2.12 The Number e

The number e is defined by infinite series and is given

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

where $n! = 1.2.3...n$ if $n \geq 1$ and $0! = 1$.

We have

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + ... + \frac{1}{1.2.3...n} \\ &< 1 + 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}} < 3, \end{aligned}$$

the series converges, and the definition makes sense.

Approximation of the number e: Let

$$s_n = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + ... + \frac{1}{1.2.3...n},$$

then we have

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + ... \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + ...\right) = \frac{1}{n!n}. \end{aligned}$$

We can conclude that

$$0 < e - s_n < \frac{1}{n!n}.$$

Approximation of the number e: We can conclude that

$$0 < e - s_n < \frac{1}{n!n}.$$

For example , take $n = 10$, thus s_{10} approximate e with an error less than 10^{-7}.

Theorem: $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$
Proof: Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad t_n = \left(1 + \frac{1}{n}\right)^n.$$

We have by using binomial theorem

$$\begin{aligned} t_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots \\ &\quad + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

the series converges, and the definition makes sense.

Hence $t_n \leq s_n$, so that

$$\lim_{n\to\infty} \sup t_n \leq e \tag{2.11}$$

Next if $n \geq m$,

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \dots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{m-1}{n}\right).$$

Let $n \to \infty$, keeping m fixed. We get

$$\lim_{n\to\infty} \inf t_n \geq 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!},$$

so that

$$s_m \leq \lim_{n} \to \infty \inf t_n.$$

Letting $m \to \infty$, we finally get

$$e \leq \lim_{n\to\infty} \inf t_n \qquad (**).$$

Theorem: The number e is an irrational number.

Proof: Before we start the proof recall the following identity

$$0 < e - s_n < \frac{1}{n!n}.$$

Suppose on contrary that e is a rational number. Then $e = p/q$, where p and q are positive integers.

Take $n = q$ then the above identity becomes

$$0 < e - s_q < \frac{1}{q!q} \quad \Rightarrow \quad 0 < q!(e - s_q) < \frac{1}{q}.$$

The number $q!e$ will be an integer (Why?)

Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!}\right)$$

is an integer, we see that $q!(e - s_q)$ is an integer. Since $q \geq 1$, the identity $0 < q!(e - s_q) < \frac{1}{q}$ implies an integer between 0 and 1.

Theorem: Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$. Then

- $\sum a_n < \infty$ if $\sum b_n < \infty$ and $\limsup_{n\to\infty} a_n/b_n < \infty$.

- $\sum a_n = \infty$ if $\sum b_n = \infty$, and $\liminf_{n\to\infty} a_n/b_n > 0$.

Proof: If $\limsup_{n\to\infty} a_n/b_n < \infty$, then $\{a_n/b_n\}$ is bounded, so there is a constant M and an integer k such that

$$a_n \leq M b_n, \quad n \geq k.$$

Since $\sum b_n < \infty$, implies that $\sum(M b_n) < \infty$. Now $\sum a_n < \infty$, by the comparison test.

If $\liminf_{n\to\infty} a_n/b_n > 0$, there is a constant m and an integer k such that

$$a_n \geq m b_n, \quad n \geq k.$$

Since $\sum b_n = \infty$, implies that $\sum(m b_n) = \infty$. Now $\sum a_n = \infty$, by the comparison test.

Theorem: Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$. Then

- $\sum a_n < \infty$ if $\sum b_n < \infty$ and $\limsup_{n\to\infty} a_n/b_n < \infty$.

- $\sum a_n = \infty$ if $\sum b_n = \infty$, and $\liminf_{n\to\infty} a_n/b_n > 0$.

Example: Let

$$\sum b_n = \sum \frac{1}{n^{p+q}} \quad \text{and} \quad \sum a_n = \sum \frac{2 + \sin n\pi/6}{(n+1)^p (n-1)^q}.$$

Then

$$\frac{a_n}{b_n} = \frac{2 + \sin n\pi/6}{(1 + 1/n)^p (1 - 1/n)^q},$$

so

$$\limsup_{n\to\infty} \frac{a_n}{b_n} = 3 \quad \text{and} \quad \liminf_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Since $\sum b_n < \infty$ if and only if $p+q>1$, the same is true of $\sum a_n$, by the previous Theorem.

Corollary: Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where $0 < L < \infty$.

Example: Let

$$\sum b_n = \sum \frac{1}{n^{p+q}}$$

and

$$\sum a_n = \sum \frac{2 + \sin n\pi/6}{(n+1)^p (n-1)^q}.$$

Example: Recall the series $\lim_{n\to\infty} \frac{1}{(n^2+n)^q}$, we have proved that the series is convergent for $q > 1/2$ by using a trick and applying comparison test.

$$\lim_{n\to\infty} \frac{1}{(n^2+n)^q} \Big/ \frac{1}{n^{2q}} = \lim_{n\to\infty} \frac{1}{(1+1/n)^q} = 1,$$

so

$$\sum \frac{1}{(n^2+n)^q} \quad \text{and} \quad \sum \frac{1}{n^{2q}}$$

converge or diverge together.

Recall the following

Theorem: Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$. Then

- $\sum a_n < \infty$ if $\sum b_n < \infty$ and $\limsup_{n\to\infty} a_n/b_n < \infty$.
- $\sum a_n = \infty$ if $\sum b_n = \infty$, and $\liminf_{n\to\infty} a_n/b_n > 0$.

2.13 The Ratio Test

Theorem: Suppose that $a_n > 0$, $b_n > 0$, and

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.$$

Then

- $\sum a_n < \infty$ if $\sum b_n < \infty$.

- $\sum b_n = \infty$ if $\sum a_n = \infty$.

Proof: We can rewrite

$$\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n},$$

we see that $\{a_n/b_n\}$ is nonincreasing. Therefore, $\limsup_{n\to\infty} a_n/b_n < \infty$, and by applying the Theorem we have proved the first part.

Theorem: Suppose that $a_n > 0$, $b_n > 0$, and

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.$$

Then

- $\sum b_n = \infty$ if $\sum a_n = \infty$.

Proof: Suppose that $\sum a_n = \infty$. Since $\{a_n/b_n\}$ is nonincreasing, there is a number ρ such that $b_n \geq \rho a_n$ for large n.

Since $\sum(\rho a_n) = \infty$ if $\sum a_n = \infty$, (with a_n replaced by ρa_n) implies that $\sum b_n = \infty$.

Example: If $\sum a_n = \sum \left(2 + \sin \frac{n\pi}{2}\right) r^n$. then

$$\frac{a_{n+1}}{a_n} = r\frac{2 + \sin \frac{(n+1)\pi}{2}}{2 + \sin \frac{n\pi}{2}}$$

which assumes the values $3r/2$, $2r/3$, $r/2$, and $2r$, each infinitely many times.

Hence,

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2r \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{r}{2}.$$

Therefore, $\sum a_n$ converges if $0 < r < 1/2$ and diverges if $r > 2$. The ratio test is inconclusive if $1/2 \leq r \leq 2$.

Corollary: Suppose that $a_n > 0$ $(n \geq k)$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then

- $\sum a_n < \infty$ if $L < 1$.
- $\sum a_n = \infty$ if $L > 1$.

The test is inconclusive if $L = 1$.

Example: Decide about the series $\sum a_n = \sum nr^{n-1}$.

Solution: Since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)r^n}{nr^{n-1}} = \left(1 + \frac{1}{n}\right) r,$$

so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

The series converges if $0 < r < 1$ and diverges if $r > 1$.

By using corollary the ratio test is inconclusive if $r = 1$. But recall that for a convergent series the nth term must approach towards zero. Hence the series diverges.

Remark: The ratio test does not imply that $\sum a_n < \infty$ if merely

$$\frac{a_{n+1}}{a_n} < 1, \tag{2.12}$$

for large n, since this could occur with $\lim_{n\to\infty} a_{n+1}/a_n = 1$, in which case the test is inconclusive.

However, the next theorem shows that $\sum a_n < \infty$ if (2.12) is replaced by the stronger condition that

$$\frac{a_{n+1}}{a_n} \le 1 - \frac{p}{n}$$

for some $p > 1$ and large n. It also shows that $\sum a_n = \infty$ if

$$\frac{a_{n+1}}{a_n} \ge 1 - \frac{q}{n}$$

for some $q < 1$ and large n.

Theorem (Raabe's test): Suppose that $a_n > 0$ for large n. Let

$$M = \limsup_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right) \quad \text{and} \quad m = \liminf_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right).$$

Then

- $\sum a_n < \infty$ if $M < -1$.
- $\sum a_n = \infty$ if $m > -1$.

The test is inconclusive if $m \le -1 \le M$.

Example: If

$$\sum a_n = \sum \frac{n!}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}, \quad \alpha > 0,$$

then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{\alpha+n} = 1,$$

so the ratio test is inconclusive.

However,

$$\begin{aligned}\lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n}-1\right) &= \lim_{n\to\infty} n\left(\frac{n+1}{\alpha+n}-1\right)\\ &= \lim_{n\to\infty} \frac{n(1-\alpha)}{\alpha+n} = 1-\alpha,\end{aligned}$$

so Raabe's test implies that $\sum a_n < \infty$ if $\alpha > 2$ and $\sum a_n = \infty$ if $0 < \alpha < 2$.

2.14 Cauchy's Root Test

Theorem: If $a_n \ge 0$ for $n \ge k$, then

- $\sum a_n < \infty$ if $\limsup_{n\to\infty} a_n^{1/n} < 1$.
- $\sum a_n = \infty$ if $\limsup_{n\to\infty} a_n^{1/n} > 1$.

The test is inconclusive if $\limsup_{n\to\infty} a_n^{1/n} = 1$.

Proof: If $\limsup_{n\to\infty} a_n^{1/n} < 1$, there is an r such that $0 < r < 1$ and $a_n^{1/n} < r$ for large n. Therefore, $a_n < r^n$ for large n. Since $\sum r^n < \infty$, the comparison test implies that $\sum a_n < \infty$.

If $\limsup_{n\to\infty} a_n^{1/n} > 1$, then $a_n^{1/n} > 1$ for infinitely many values of n, so $\sum a_n = \infty$.

Example: Cauchy's root test is inconclusive if

$$\sum a_n = \sum \frac{1}{n^p},$$

because then

$$\limsup_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty} \exp\left(-\frac{p}{n}\log n\right) = 1$$

for all p.

However, we know that $\sum 1/n^p < \infty$ if $p > 1$ and $\sum 1/n^p = \infty$ if $p \le 1$.

Example: If

$$\sum a_n = \sum \left(2+\sin\frac{n\pi}{4}\right)^n r^n,$$

then

$$\limsup_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} \left(2+\sin\frac{n\pi}{4}\right) r = 3r,$$

and so $\sum a_n < \infty$ if $r < 1/3$ and $\sum a_n = \infty$ if $r > 1/3$. The test is inconclusive if $r = 1/3$, but then $|a_{8m+2}| = 1$ for $m \ge 0$, so $\sum a_n = \infty$.

2.15 Absolute Convergence

2.16 Absolute Convergence

A series $\sum a_n$ *converges absolutely*, or is *absolutely convergent*, if $\sum |a_n| < \infty$.

Example: A convergent series $\sum a_n$ of nonnegative terms is absolutely convergent, since $\sum a_n$ and $\sum |a_n|$ are the same.

More generally, any convergent series whose terms are of the same sign for sufficiently large n converges absolutely.

Example: Consider the series
$$\sum \frac{\sin n\theta}{n^p},$$
where θ is arbitrary and $p > 1$.

Solution: Since
$$\left|\frac{\sin n\theta}{n^p}\right| \le \frac{1}{n^p}$$
and $\sum 1/n^p < \infty$ if $p > 1$, the comparison test implies that
$$\sum \left|\frac{\sin n\theta}{n^p}\right| < \infty, \quad p > 1.$$
Therefore, the given series converges absolutely if $p > 1$.

Example: If $0 < p < 1$, then the series
$$\sum \frac{(-1)^n}{n^p}$$
does not converge absolutely, since
$$\sum \left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p} = \infty.$$
However, the series converges, by the alternating series test, which we prove below.

Remark: Any test for convergence of a series with nonnegative terms can be used to test an arbitrary series $\sum a_n$ for absolute convergence by applying it to $\sum |a_n|$.

Example: Determine the series
$$\sum a_n = \sum (-1)^n \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n-1)}, \quad \alpha > 0,$$
for absolutely convergent or not?

We apply Raabe's test to

$$\sum a_n = \sum \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n-1)}.$$

From previous example, $\sum |a_n| < \infty$ if $\alpha > 2$ and $\sum |a_n| = \infty$ if $\alpha < 2$. Therefore, $\sum a_n$ converges absolutely if $\alpha > 2$, but not if $\alpha < 2$.

Notice that this does not imply that $\sum a_n$ diverges if $\alpha < 2$.

Theorem: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof: See Lecture.

Example: For example, the Theorem implies that

$$\sum \frac{\sin n\theta}{n^p}$$

converges if $p > 1$, since it then converges absolutely.

What about the converse of the theorem?

Conditional convergence

2.17 Dirichlet's Test for Series

Theorem: The series $\sum_{n=k}^{\infty} a_n b_n$ converges if $\lim_{n\to\infty} a_n = 0$,

$$\sum |a_{n+1} - a_n| < \infty,$$

and

$$|b_k + b_{k+1} + \cdots + b_n| \le M, \quad n \ge k,$$

for some constant M.

Proof: Define $B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \ge k$ and consider the partial sums of $\sum_{n=k}^{\infty} a_n b_n$:

$$S_n = a_k b_k + a_{k+1} b_{k+1} + \cdots + a_n b_n, \quad n \ge k. \tag{2.13}$$

By substituting

$$b_k = B_k \quad \text{and} \quad b_n = B_n - B_{n-1}, \quad n \ge k+1,$$

into (2.13), we obtain

$$S_n = a_k B_k + a_{k+1}(B_{k+1} - B_k) + \cdots + a_n(B_n - B_{n-1}).$$

Rewriting as

$$\begin{aligned} S_n \;=\; & (a_k - a_{k+1})B_k + (a_{k+1} - a_{k+2})B_{k+1} + \cdots \\ & + (a_{n-1} - a_n)B_{n-1} + a_n B_n. \end{aligned} \tag{2.14}$$

(The procedure that led from (2.13) to (2.14) is called *summation by parts*. It is analogous to integration by parts.)

Now (2.14) can be viewed as

$$S_n = T_{n-1} + a_n B_n, \tag{2.15}$$

where

$$T_{n-1} = (a_k - a_{k+1})B_k + (a_{k+1} - a_{k+2})B_{k+1} + \cdots + (a_{n-1} - a_n)B_{n-1};$$

that is, $\{T_n\}$ is the sequence of partial sums of the series

$$\sum_{j=k}^{\infty}(a_j - a_{j+1})B_j. \tag{2.16}$$

Since

$$|(a_j - a_{j+1})B_j| \leq M|a_j - a_{j+1}|$$

from given conditions and the comparison test imply that the series (2.16) converges absolutely. Absolutely convergence implies that $\{T_n\}$ converges.

Let $T = \lim_{n\to\infty} T_n$. Since $\{B_n\}$ is bounded and $\lim_{n\to\infty} a_n = 0$, we infer from (2.15) that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_{n-1} + \lim_{n\to\infty} a_n B_n = T + 0 = T.$$

Therefore, $\sum a_n b_n$ converges.

Example: Apply Dirichlet's test to the following series

$$\sum_{n=2}^{\infty} \frac{\sin n\theta}{n + (-1)^n}, \quad \theta \neq k\pi \quad (k = \text{integer}),$$

Solution: We take

$$a_n = \frac{1}{n + (-1)^n} \quad \text{and} \quad b_n = \sin n\theta.$$

Then $\lim_{n\to\infty} a_n = 0$, and

$$|a_{n+1} - a_n| < \frac{3}{n(n-1)}$$

(verify), so

$$\sum |a_{n+1} - a_n| < \infty.$$

Now

$$B_n = \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta.$$

We can write

$$B_n = \frac{(\cos\frac{3}{2}\theta - \cos\frac{5}{2}\theta) + + \cdots + \left(\cos\left(n-\frac{1}{2}\right)\theta - \cos(n+\frac{1}{2})\theta\right)}{2\sin(\theta/2)}$$

$$= \frac{\cos\frac{3}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin(\theta/2)},$$

which implies that $|B_n| \leq \left|\frac{1}{\sin(\theta/2)}\right|, \quad n \geq 2.$

Since $\{a_n\}$ and $\{b_n\}$ satisfy the hypotheses of Dirichlet's theorem, $\sum a_n b_n$ converges.

Remark: Dirichlet's test takes a simpler form if $\{a_n\}$ is nonincreasing, as follows.

Corollary: The series $\sum a_n b_n$ converges if $a_{n+1} \leq a_n$ for $n \geq k$, $\lim_{n\to\infty} a_n = 0$, and

$$|b_k + b_{k+1} + \cdots + b_n| \leq M, \quad n \geq k,$$

for some constant M.

Proof: Recall the Dirichlet's test for series, we need to show that

$$\sum_{n=k}^{\infty} |a_{n+1} - a_n| = a_k < \infty.$$

If $a_{n+1} \leq a_n$, then $\sum_{n=k}^{m} |a_{n+1} - a_n| = \sum_{n=k}^{m} (a_n - a_{n+1}) = a_k - a_{m+1}$.
Since $\lim_{m\to\infty} a_{m+1} = 0$, it follows that

$$\sum_{n=k}^{\infty} |a_{n+1} - a_n| = a_k < \infty.$$

Therefore, the hypotheses of Dirichlet's test are satisfied, so $\sum a_n b_n$ converges.

Example: Discuss the convergence of the series

$$\sum \frac{\sin n\theta}{n^p}.$$

Solution: The given series is convergent if $p > 1$.

If we take $a_n = 1/n^p$ and $b_n = \sin n\theta$, then we have

$$a_{n+1} \leq a_n, \qquad \lim_{n\to\infty} = 0.$$

and we have proved in previous example that

$$|b_k + b_{k+1} + \cdots + b_n| \leq M, \quad n \geq k,$$

Consequently, the given series also converges if $0 < p \leq 1$ and this follows from Abel's test.

2.18 Alternating Series

Series whose terms alternate between positive and negative, called alternating series.

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$

$$\sum_{k=1}^{\infty}(-1)^{k}\frac{1}{k}.$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k, \qquad \sum_{k=1}^{\infty}(-1)^{k+1}a_k.$$

Corollary (Alternating series test): The series $\sum(-1)^n a_n$ converges if $0 \leq a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$.

Proof: Let $b_n = (-1)^n$; recall that

$$B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \geq k.$$

Then $\{|B_n|\}$ is a sequence of zeros and ones and therefore bounded. The conclusion now follows from Abel's test.

Example: Show that the series is convergent

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$

Example: Show that the series is convergent

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}.$$

2.19 Grouping Terms in a Series

- The terms of a finite sum can be grouped by putting parenthesis arbitrarily.
- For example,

$$\begin{aligned}&(1+7)+(6+5)+4 =\\&(1+7+6)+(5+4).\end{aligned}$$

- Is the same true for infinite series?

Theorem: Suppose that $\sum_{n=k}^{\infty} a_n = A$, where $-\infty \leq A \leq \infty$. Let $\{n_j\}_1^{\infty}$ be an increasing sequence of integers, with $n_1 \geq k$.

Define

$$\begin{aligned} b_1 &= a_k + \cdots + a_{n_1}, \\ b_2 &= a_{n_1+1} + \cdots + a_{n_2}, \\ &\vdots \\ b_r &= a_{n_{r-1}+1} + \cdots + a_{n_r}. \end{aligned}$$

Then

$$\sum_{j=1}^{\infty} b_{n_j} = A.$$

Proof: If T_r is the rth partial sum of $\sum_{j=1}^{\infty} b_{n_j}$ and $\{A_n\}$ is the nth partial sum of $\sum_{s=k}^{\infty} a_s$.

Then

$$\begin{aligned} T_r &= b_1 + b_2 + \cdots + b_r \\ &= (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots + \\ &\quad (a_{n_{r-1}+1} + \cdots + a_{n_r}) \\ &= A_{n_r}. \end{aligned}$$

Thus, $\{T_r\}$ is a subsequence of $\{A_n\}$, so

$$\lim_{r\to\infty} T_r = \lim_{n\to\infty} A_n = A.$$

Example: If $\sum_{n=0}^{\infty}(-1)^n a_n$ satisfies the hypotheses of the alternating series test and converges to the sum S

Then the theorem of grouping terms in a series enables us to write

$$S = \sum_{n=0}^{k}(-1)^n a_n + (-1)^{k+1}\sum_{j=1}^{\infty}(a_{k+2j-1} - a_{k+2j})$$

and

$$S = \sum_{n=0}^{k}(-1)^n a_n + (-1)^{k+1}\left[a_{k+1} - \sum_{j=1}^{\infty}(a_{k+2j} - a_{k+2j-1})\right].$$

Since $0 \leq a_{n+1} \leq a_n$, these two equations imply that $S - S_k$ is between 0 and $(-1)^{k-1}a_{k+1}$.

- Be careful while introducing parenthesis to a divergent series
- Apply carefully the previous theorem

Example: For example, it is tempting to write

$$\sum_{n=1}^{\infty}(-1)^{n+1} = (1-1)+(1-1)+\cdots = 0+0+\cdots$$

and conclude that $\sum_{n=1}^{\infty}(-1)^n = 0$.

But equally tempting to write

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1} &= 1-(1-1)-(1-1)-\cdots \\ &= 1-0-0-\cdots\end{aligned}$$

and conclude that $\sum_{n=1}^{\infty}(-1)^{n+1} = 1$.

Is there a contradiction with the theorem? Of course, there is no contradiction here, since Theorem does not apply to this series, and neither of these operations is legitimate.

2.20 Rearrangements of Series

Theorem: If $\sum_{n=1}^{\infty} b_n$ is a rearrangement of an absolutely convergent series $\sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ also converges absolutely, and to the same sum.

Proof: Let

$$\overline{A}_n = |a_1|+|a_2|+\cdots+|a_n| \quad \text{and} \quad \overline{B}_n = |b_1|+|b_2|+\cdots+|b_n|.$$

For each $n \geq 1$, there is an integer k_n such that $b_1, b_2, \ldots, b_n$ are included among $a_1, a_2, \ldots, a_{k_n}$, so $\overline{B}_n \leq \overline{A}_{k_n}$. Since $\{\overline{A}_n\}$ is bounded, so is $\{\overline{B}_n\}$, and therefore $\sum |b_n| < \infty$.

Now let

$$\begin{aligned} A_n &= a_1+a_2+\cdots+a_n, \quad B_n = b_1+b_2+\cdots+b_n, \\ A &= \sum_{n=1}^{\infty} a_n, \quad \text{and} \quad B = \sum_{n=1}^{\infty} b_n. \end{aligned}$$

We must show that $A = B$. Suppose that $\varepsilon > 0$. From Cauchy's convergence criterion for series and the absolute convergence of $\sum a_n$, there is an integer N such that

$$|a_{N+1}|+|a_{N+2}|+\cdots+|a_{N+k}| < \varepsilon, \quad k \geq 1.$$

Choose N_1 so that a_1, a_2, ..., a_N are included among b_1, b_2, ..., b_{N_1}. If $n \geq N_1$, then A_n and B_n both include the terms a_1, a_2, ..., a_N, which cancel on subtraction; thus, $|A_n - B_n|$ is dominated by the sum of the absolute values of finitely many terms from $\sum a_n$ with subscripts greater than N.

Since every such sum is less than ε,

$$|A_n - B_n| < \varepsilon \quad \text{if} \quad n \geq N_1.$$

Therefore, $\lim_{n\to\infty}(A_n - B_n) = 0$ and $A = B$.

2.21 Addition and Multiplication of Series

Theorem (Addition of series): If $\sum a_n = A$ and $\sum b_n = B$; then their sum $\sum(a_n + b_n) = A + B$ and $\sum ca_n = cA$, for some fixed c.

Proof: Consider the sequence of partial sums

Product of infinite series: Given two series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n.$$

We can arrange all possible products $a_i b_j$ $(i, j \geq 0)$ in a two-dimensional array:

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array} \tag{2.17}$$

where the subscript on a is constant in each row and the subscript on b is constant in each column.

Any sensible definition of the product

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$$

clearly must involve every product in this array exactly once; thus, we might define the product of the two series to be the series $\sum_{n=0}^{\infty} p_n$, where $\{p_n\}$ is a sequence obtained by ordering the products in (2.17) according to some method that chooses every product exactly once. One way to do this is indicated by

$$
\begin{array}{ccccccccc}
a_0b_0 & \rightarrow & a_0b_1 & & a_0b_2 & \rightarrow & a_0b_3 & \cdots \\
 & & \downarrow & & \uparrow & & \downarrow & \\
a_1b_0 & \leftarrow & a_1b_1 & & a_1b_2 & & a_1b_3 & \cdots \\
\downarrow & & & & \uparrow & & \downarrow & \\
a_2b_0 & \rightarrow & a_2b_1 & \rightarrow & a_2b_2 & & a_2b_3 & \cdots \\
 & & & & & & \downarrow & \\
a_3b_0 & \leftarrow & a_3b_1 & \leftarrow & a_3b_2 & \leftarrow & a_3b_3 & \cdots \\
\downarrow & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
\tag{2.18}
$$

Product of infinite series: Another by

$$
\begin{array}{cccccccccc}
a_0b_0 & \rightarrow & a_0b_1 & & a_0b_2 & \rightarrow & a_0b_3 & & a_0b_4 & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \\
a_1b_0 & & a_1b_1 & & a_1b_2 & & a_1b_3 & & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & \\
a_2b_0 & & a_2b_1 & & a_2b_2 & & a_2b_3 & & \cdots & \\
 & \swarrow & & \nearrow & & & & & & \\
a_3b_0 & & a_3b_1 & & a_3b_2 & & a_3b_3 & & \cdots & \\
\downarrow & \nearrow & & & & & & & & \\
a_4b_0 & & \vdots & & \vdots & & \vdots & & &
\end{array}
\tag{2.19}
$$

There are infinitely many others, and to each corresponds a series that we might consider to be the product of the given series. This raises a question: If

$$\sum_{n=0}^{\infty} a_n = A \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = B.$$

where A and B are finite, does every product series $\sum_{n=0}^{\infty} p_n$ constructed by ordering the products in (2.17) converge to AB?

The next theorem tells us when the answer is yes.

Theorem: Let $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$,

where A and B are finite, and at least one term of each series is nonzero. Then $\sum_{n=0}^{\infty} p_n = AB$ for every sequence $\{p_n\}$ obtained by ordering the products if and only if $\sum a_n$ and $\sum b_n$ converge absolutely. Moreover, in this case, $\sum p_n$ converges absolutely.

Proof: Before we start the proof let us recall one way of writing the product is

$$
\begin{array}{ccccccccc}
a_0b_0 & \to & a_0b_1 & & a_0b_2 & \to & a_0b_3 & \cdots \\
 & & \downarrow & & \uparrow & & \downarrow & \\
a_1b_0 & \leftarrow & a_1b_1 & & a_1b_2 & & a_1b_3 & \cdots \\
\downarrow & & & & \uparrow & & \downarrow & \\
a_2b_0 & \to & a_2b_1 & \to & a_2b_2 & & a_2b_3 & \cdots \\
 & & & & & & \downarrow & \\
a_3b_0 & \leftarrow & a_3b_1 & \leftarrow & a_3b_2 & \leftarrow & a_3b_3 & \cdots \\
\downarrow & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
\tag{2.20}
$$

First, let $\{p_n\}$ be the sequence obtained by arranging the products $\{a_ib_j\}$ according to the scheme indicated just above, and define

$$
\begin{aligned}
A_n &= a_0 + a_1 + \cdots + a_n, \quad & \overline{A}_n &= |a_0| + |a_1| + \cdots + |a_n|, \\
B_n &= b_0 + b_1 + \cdots + b_n, \quad & \overline{B}_n &= |b_0| + |b_1| + \cdots + |b_n|, \\
P_n &= p_0 + p_1 + \cdots + p_n, \quad & \overline{P}_n &= |p_0| + |p_1| + \cdots + |p_n|.
\end{aligned}
$$

From arrangement, we see that

$$P_0 = A_0B_0, \quad P_3 = A_1B_1, \quad P_8 = A_2B_2,$$

and, in general,

$$P_{(m+1)^2-1} = A_mB_m. \tag{2.21}$$

Similarly,

$$\overline{P}_{(m+1)^2-1} = \overline{A}_m\overline{B}_m. \tag{2.22}$$

If $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$, then $\{\overline{A}_m\overline{B}_m\}$ is bounded.

Since $\overline{P}_m \leq \overline{P}_{(m+1)^2-1}$, (2.22) implies that $\{\overline{P}_m\}$ is bounded.

Therefore, $\sum |p_n| < \infty$, so $\sum p_n$ converges. Now

$$
\begin{aligned}
\textstyle\sum_{n=0}^{\infty} p_n &= \lim_{n\to\infty} P_n && \text{(by definition)} \\
&= \lim_{m\to\infty} P_{(m+1)^2-1} \\
&= \lim_{m\to\infty} A_mB_m && \text{(from (2.21))} \\
&= \left(\lim_{m\to\infty} A_m\right)\left(\lim_{m\to\infty} B_m\right) \\
&= AB.
\end{aligned}
$$

Since any other ordering of the products produces a rearrangement of the absolutely convergent series $\sum_{n=0}^{\infty} p_n$.

Theorem about rearrangements of absolutely convergent series implies that $\sum |q_n| < \infty$ for every such ordering and that $\sum_{n=0}^{\infty} q_n = AB$.

This shows that the stated condition is sufficient.

For necessity, again let $\sum_{n=0}^{\infty} p_n$ be obtained from the ordering indicated in (2.20). Suppose that $\sum_{n=0}^{\infty} p_n$ and all its rearrangements converge to AB. Then $\sum p_n$ must converge absolutely (Why).

Therefore, $\{\overline{P}_{m^2-1}\}$ is bounded, and (2.22) implies that $\{\overline{A}_m\}$ and $\{\overline{B}_m\}$ are bounded.

(Here we need the assumption that neither $\sum a_n$ nor $\sum b_n$ consists entirely of zeros. Why?) Therefore, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$.

2.22 Power Series

Maclaurin and Taylor series: If f has derivatives of all order at x_0, then we call the series

$$\begin{aligned}\sum_{n=0}^{\infty}(x-x_0)^n &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...\\ &\quad +\frac{f^n(x_0)}{n!}(x-x_0)^n+...\end{aligned}$$

is known as Taylor series for f at the point $x = x_0$.

The special case of Taylor series when $x_0 = 0$, the series is known as Maclaurin series

$$\sum_{n=0}^{\infty} x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+...+\frac{f^n(0)}{k!}x^n+....$$

Examples:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}+....$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-...+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+....$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...+(-1)^n\frac{x^{2n}}{(2n)!}+....$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+...+x^n+....$$

Power Series: An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

where x_0 and a_0, a_1, ..., are constants, is called a *power series in* $x - x_0$.

If $x_0 = 0$ then power series becomes

$$\sum_{n=0}^{\infty} a_n x^n.$$

Theorem: For the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, define R in the extended real numbers by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

In particular, $R = 0$, $R = \infty$ or $0 < R < \infty$. Then the power series converges

- If $\limsup_{n\to\infty} |a_n|^{1/n} = \infty$ then $R = 0$ and the power series converges only for $x = x_0$.
- If $\limsup_{n\to\infty} |a_n|^{1/n} = 0$, then $R = \infty$ and the power series converges for all x.
- If $\limsup_{n\to\infty} |a_n|^{1/n} \neq 0$ such that $0 < R < \infty$, then power series converges for x in $(x_0 - R, x_0 + R)$.
- The series diverges if $|x - x_0| > R$.
- No general statement can be made concerning convergence at the endpoints $x = x_0+R$ and $x = x_0-R$: the series may converge absolutely or conditionally at both, converge conditionally at one and diverge at the other, or diverge at both.

Theorem: For the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, define R in the extended real numbers by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Proof: Put $y = x - x_0$, $c_n = a_n y^n$ and apply the root test

$$\limsup_{n\to\infty} |c_n|^{1/n} = |y| \limsup_{n\to\infty} |a_n|^{1/n} = \frac{|y|}{R}.$$

Remark: R is called the radius of convergence of the power series and $(x_0 - R, x_0 + R)$ is known as interval of convergence.

Example: For the power series

$$\sum \frac{\sin n\pi/6}{2^n}(x - 1)^n,$$

what is a_n, x_0 and R?

Solution: For R, we have

$$\begin{aligned}
\limsup_{n\to\infty} |a_n|^{1/n} &= \limsup_{n\to\infty} \left(\frac{|\sin n\pi/6}{2^n}\right)^{1/n} \\
&= \frac{1}{2}\limsup_{n\to\infty}(|\sin n\pi/6|)^{1/n} \\
&= \frac{1}{2}(1) = \frac{1}{2}.
\end{aligned}$$

Therefore, $R = 2$ and Theorem about the convergence of power series implies that the series converges absolutely uniformly in closed subintervals of $(-1, 3)$ and diverges if $x < -1$ or $x > 3$.

Remark: The Theorem about the convergence of the power series does not tell us what happens when $x = -1$ or $x = 3$. we can see that the series diverges in both these cases since its general term does not approach zero.

Example: For the power series

$$\sum \frac{x^n}{n},$$

what is a_n, x_0 and R?

Solution: For R, we have

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \limsup_{n\to\infty} \exp\left(\frac{1}{n}\log\frac{1}{n}\right) = e^0 = 1.$$

Therefore, $R = 1$ and the series converges absolutely uniformly in closed subintervals of $(-1, 1)$ and diverges if $|x| > 1$.

For $x = -1$ the series becomes $\sum(-1)^n/n$, which converges conditionally, and at $x = 1$ the series becomes $\sum 1/n$, which diverges.

Theorem: The radius of convergence of $\sum a_n(x - x_0)^n$ is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

if the limit exists in the extended real number system.

Proof: It is sufficient to show that if

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

exists in the extended real number system, then

$$L = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Let $0 < L < \infty$, if

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

holds with $0 < L < \infty$ and $0 < \varepsilon < L$, there is an integer N such that

$$L - \varepsilon < \left| \frac{a_{m+1}}{a_m} \right| < L + \varepsilon \quad \text{if} \quad m \geq N,$$

so

$$|a_m|(L - \varepsilon) < |a_{m+1}| < |a_m|(L + \varepsilon) \quad \text{if} \quad m \geq N.$$

By induction,

$$|a_N|(L - \varepsilon)^{n-N} < |a_n| < |a_N|(L + \varepsilon)^{n-N} \quad \text{if} \quad n > N.$$

Therefore, if

$$K_1 = |a_N|(L - \varepsilon)^{-N} \quad \text{and} \quad K_2 = |a_N|(L + \varepsilon)^{-N},$$

then

$$K_1^{1/n}(L - \varepsilon) < |a_n|^{1/n} < K_2^{1/n}(L + \varepsilon). \tag{2.23}$$

Since $\lim_{n\to\infty} K^{1/n} = 1$ if K is any positive number, (2.23) implies that

$$L - \varepsilon \leq \liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq L + \varepsilon.$$

Since ε is an arbitrary positive number, it follows that

$$\lim_{n\to\infty} |a_n|^{1/n} = L,$$

hence the proof.

Example: Determine the radius of convergence for the power series

$$\sum \frac{x^n}{n!}.$$

Solution: We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Therefore, $R = \infty$; that is, the series converges for all x.

Example: Determine the radius and interval of convergence for the power series

$$\sum n! x^n.$$

Solution:We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty.$$

Therefore, $R = 0$, and the series converges only if $x = 0$.

Example: Find the interval of convergence for the power series

$$\sum \frac{(-1)^n}{4^n n^p} x^{2n} \quad (p = \text{constant}), \tag{2.24}$$

Solution: The given power series has infinitely many zero coefficients (of odd powers of x).

However, by setting $y = x^2$, we obtain the series

$$\sum \frac{(-1)^n}{4^n n^p} y^n, \tag{2.25}$$

which has nonzero coefficients for which

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4^n n^p}{4^{n+1}(n+1)^p} = \frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-p} = \frac{1}{4}.$$

Therefore, (2.25) converges if $|y| < 4$ and diverges if $|y| > 4$.

Setting $y = x^2$, we conclude that (2.24) converges if $|x| < 2$ and diverges if $|x| > 2$.

Chapter 3

Continuity

Function: A rule f that assigns to each member of a nonempty set D a unique member of a set Y is a *function from D to Y*. The relationship between a member x of D and the member y of Y that f assigns to x as

$$y = f(x).$$

The set D is the *domain* of f, denoted by D_f. The members of Y are the possible *values* of f.

If $y_0 \in Y$ and there is an x_0 in D such that $f(x_0) = y_0$ then we say that f *attains* or *assumes* the value y_0. The set of values attained by f is the *range* of f.

A *real-valued function of a real variable* is a function whose domain and range are both subsets of the real numbers.

Examples: Consider the functions

$$f(x) = x^2, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = e^x.$$

The functions f, g, and h defined on the extended real number system $(-\infty, \infty)$.

- The range of f is $[0, \infty)$.
- The range of g is $[-1, 1]$.
- The range of h is $(0, \infty)$.

The equation

$$[f(x)]^2 = x$$

does not define a function except on the singleton set $\{0\}$. If $x < 0$, no real number satisfies the $[f(x)]^2 = x$, while if $x > 0$, two real numbers satisfy $[f(x)]^2 = x$. However, the conditions

$$[f(x)]^2 = x \quad \text{and} \quad f(x) \geq 0$$

define a function f on $D_f = [0, \infty)$ with values $f(x) = \sqrt{x}$.

Similarly, the conditions

$$[g(x)]^2 = x \quad \text{and} \quad g(x) \leq 0$$

define a function g on $D_g = [0, \infty)$ with values $g(x) = -\sqrt{x}$. The ranges of f and g are $[0, \infty)$ and $(-\infty, 0]$, respectively.

We first define the *Cartesian product* $X \times Y$ of two nonempty sets X and Y to be the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$; thus,

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

If (x, y) and (x, y_1) are in f, then $y = y_1$. The set of x's that occur as first members of f is the of f.

If x is in the domain of f, then the unique y in Y such that $(x, y) \in f$ is the *value of f at x*, and we write $y = f(x)$.

The set of all such values, a subset of Y, is the *range of f*.

If $D_f \cap D_g \neq \emptyset$, then $f + g$, $f - g$, and fg are defined on $D_f \cap D_g$ by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (f-g)(x) &= f(x) - g(x), \end{aligned}$$

and

$$(fg)(x) = f(x)g(x).$$

The quotient f/g is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

for x in $D_f \cap D_g$ such that $g(x) \neq 0$.

Example: Define $(f+g)(x), (f-g)(x), (fg)(x)$ and (f/g), for the functions $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x - 1}$.

Solution: First of all observe that $D_f = [-2, 2]$ and $D_g = [1, \infty)$.

Then $f + g$, $f - g$, and fg are defined on $D_f \cap D_g = [1, 2]$ by

$$\begin{aligned} (f+g)(x) &= \sqrt{4 - x^2} + \sqrt{x - 1}, \\ (f-g)(x) &= \sqrt{4 - x^2} - \sqrt{x - 1}, \end{aligned}$$

and

$$(fg)(x) = (\sqrt{4 - x^2})(\sqrt{x - 1}) = \sqrt{(4 - x^2)(x - 1)}.$$

The quotient f/g is defined on $(1, 2]$ by

$$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{4 - x^2}{x - 1}}.$$

Although the last expression in (**??**) is also defined for $-\infty < x < -2$, it does not represent fg for such x, since f and g are not defined on $(-\infty, -2]$.

3.1 Limits

The tangent line problem: Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P.

The area problem: Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis.

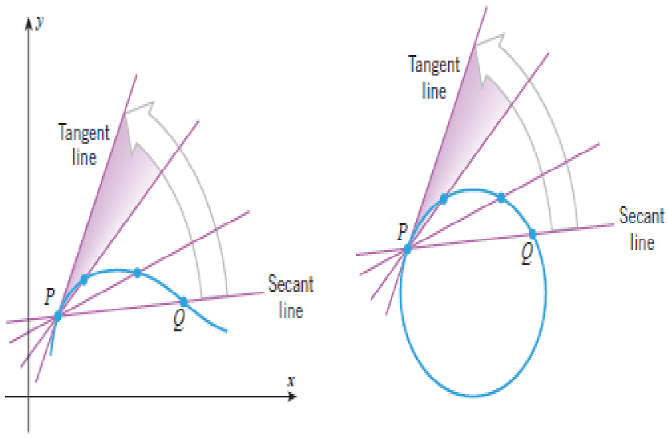


Figure 3.1: Tangent line as a limit of secant line

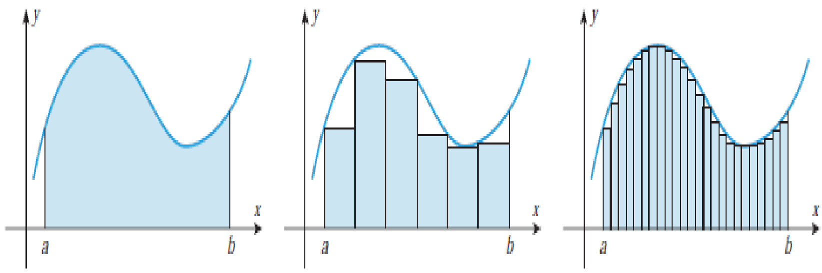


Figure 3.2: Area under the curve as limit

Example: Examine the behavior of the function $f(x) = x^2 - x + 1$ for x values closer to 2.

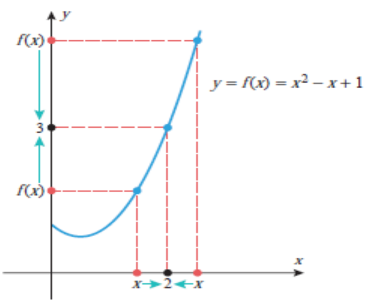


Figure 3.3: Limit $x \to 2$

3.1.1 Limits (An informal view)

If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L.$$

Sometimes it is also written as

$$f(x) \to L, \qquad \text{as} \qquad x \to a.$$

Example: Investigate the limit

$$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}.$$

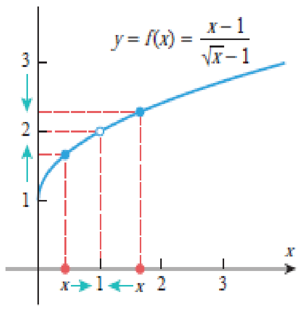


Figure 3.4: Limit $x \to 1$ when $f(x)$ is not defined at $x = 1$

Example: Investigate the limit

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

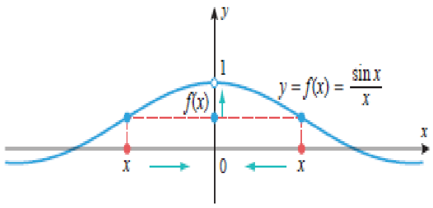


Figure 3.5: Limit $x \to 0$ when $f(x)$ is not defined at $x = 0$

Example: Investigate the limit

$$\lim_{x\to 0} \frac{|x|}{x}.$$

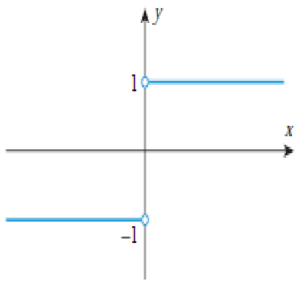


Figure 3.6: Limit $x \to 0$

3.1.2 Formal Definition of Limit

We say that $f(x)$ *approaches the limit* L *as* x *approaches* x_0, and write

$$\lim_{x\to x_0} f(x) = L,$$

if f is defined on some deleted neighborhood of x_0 and, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \qquad \text{whenever} \qquad 0 < |x - x_0| < \delta. \tag{3.1}$$

Example: For the function defined by $f(x) = cx$, where $c \in \mathbb{R}$. Show that

$$\lim_{x\to x_0} f(x) = cx_0.$$

Solution: We write

$$|f(x) - cx_0| = |cx - cx_0| = |c||x - x_0|.$$

If $c \neq 0$, this yields

$$|f(x) - cx_0| < \varepsilon$$

if

$$|x - x_0| < \delta,$$

where δ is any number such that $0 < \delta \leq \varepsilon/|c|$.

If $c = 0$, then $f(x) - cx_0 = 0$ for all x, so $|f(x) - cx_0| < \varepsilon$ holds for all x.

Example: Prove that $\lim_{x\to 2}(3x - 5) = 1$.

Solution: See Lecture.

Example: For the function

$$f(x) = x \sin \frac{1}{x}, \quad x \neq 0,$$

show that

$$\lim_{x\to 0} f(x) = 0.$$

Solution: Even though f is not defined at $x_0 = 0$, because if

$$0 < |x| < \delta = \varepsilon,$$

then

$$|f(x) - 0| = \left| x \sin \frac{1}{x} \right| \leq |x| < \varepsilon.$$

On the other hand, the function

$$g(x) = \sin \frac{1}{x}, \quad x \neq 0,$$

has no limit as x approaches 0, since it assumes all values between -1 and 1 in every neighborhood of the origin.

Theorem: If $\lim_{x\to x_0} f(x)$ exists, then it is unique.

Proof: Suppose that limit of the function exists and it is L_1 and L_2. Let $\varepsilon > 0$. From definition of the limit, there are positive numbers δ_1 and δ_2 such that

$$|f(x) - L_i| < \varepsilon \quad \text{if} \quad 0 < |x - x_0| < \delta_i, \quad i = 1, 2.$$

If $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned}|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |L_1 - f(x)| + |f(x) - L_2| < 2\varepsilon \quad \text{if} \quad 0 < |x - x_0| < \delta.\end{aligned}$$

We have now established an inequality that does not depend on x; that is,

$$|L_1 - L_2| < 2\varepsilon.$$

Since this holds for any positive ε, $L_1 = L_2$.

Theorem: If

$$\lim_{x\to x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x\to x_0} g(x) = L_2. \tag{3.2}$$

then

$$\begin{aligned}\lim_{x\to x_0} (f+g)(x) &= L_1 + L_2, \\ \lim_{x\to x_0} (f-g)(x) &= L_1 - L_2, \\ \lim_{x\to x_0} (fg)(x) &= L_1 L_2, \\ \lim_{x\to x_0} \left(\frac{f}{g}\right)(x) &= \frac{L_1}{L_2}, \qquad L_2 \ne 0.\end{aligned}$$

Proof: From (3.2) and definition of the limit, if $\varepsilon > 0$, there is a $\delta_1 > 0$ such that

$$|f(x) - L_1| < \varepsilon \tag{3.3}$$

if $0 < |x - x_0| < \delta_1$, and a $\delta_2 > 0$ such that

$$|g(x) - L_2| < \varepsilon \tag{3.4}$$

if $0 < |x - x_0| < \delta_2$. Suppose that

$$0 < |x - x_0| < \delta = \min(\delta_1, \delta_2), \tag{3.5}$$

so that (3.3) and (3.4) both hold. Then

$$\begin{aligned}|(f \pm g)(x) - (L_1 \pm L_2)| &= |(f(x) - L_1) \pm (g(x) - L_2)| \\ &\le |f(x) - L_1| + |g(x) - L_2| < 2\varepsilon,\end{aligned}$$

which proves (3.3) and (3.3).

We write

$$\begin{aligned}|(fg)(x) - L_1L_2| &= |f(x)g(x) - L_1L_2| \\ &= |f(x)(g(x) - L_2) + L_2(f(x) - L_1)| \\ &\le |f(x)||g(x) - L_2| + |L_2||f(x) - L_1| \\ &\le (|f(x)| + |L_2|)\varepsilon \\ &\le (|f(x) - L_1| + |L_1| + |L_2|)\varepsilon \\ &\le (\varepsilon + |L_1| + |L_2|) \\ &\le (1 + |L_1| + |L_2|)\varepsilon\end{aligned}$$

First observe that if $L_2 \neq 0$, there is a $\delta_3 > 0$ such that

$$|g(x) - L_2| < \frac{|L_2|}{2}, \qquad |g(x)| > \frac{|L_2|}{2} \quad \text{if} \quad 0 < |x - x_0| < \delta_3.$$

Now suppose that $0 < |x - x_0| < \min(\delta_1, \delta_2, \delta_3)$. Then

$$\begin{aligned}
\left|\left(\frac{f}{g}\right)(x) - \frac{L_1}{L_2}\right| &= \left|\frac{f(x)}{g(x)} - \frac{L_1}{L_2}\right| = \frac{|L_2 f(x) - L_1 g(x)|}{|g(x) L_2|} \\
&\leq \frac{2}{|L_2|^2} |L_2 f(x) - L_1 g(x)| \\
&= \frac{2}{|L_2|^2} \left|L_2[f(x) - L_1] + L_1[L_2 - g(x)]\right| \\
&\leq \frac{2}{|L_2|^2} \left[|L_2||f(x) - L_1| + |L_1||L_2 - g(x)|\right] \\
&\leq \frac{2}{|L_2|^2} (|L_2| + |L_1|)\varepsilon.
\end{aligned}$$

Example: Find

$$\lim_{x \to 2} \frac{9 - x^2}{x + 1} \quad \text{and} \quad \lim_{x \to 2} (9 - x^2)(x + 1).$$

Solution: If c is a constant, then $\lim_{x \to x_0} c = c$, and $\lim_{x \to x_0} x = x_0$. Therefore

$$\begin{aligned}
\lim_{x \to 2} (9 - x^2) &= \lim_{x \to 2} 9 - \lim_{x \to 2} x^2 \\
&= \lim_{x \to 2} 9 - (\lim_{x \to 2} x)^2 \\
&= 9 - 2^2 = 5,
\end{aligned}$$

and

$$\lim_{x \to 2} (x + 1) = \lim_{x \to 2} x + \lim_{x \to 2} 1 = 2 + 1 = 3.$$

Therefore,

$$\lim_{x \to 2} \frac{9 - x^2}{x + 1} = \frac{\lim_{x \to 2} (9 - x^2)}{\lim_{x \to 2} (x + 1)} = \frac{5}{3}$$

and

$$\lim_{x \to 2} (9 - x^2)(x + 1) = \lim_{x \to 2} (9 - x^2) \lim_{x \to 2} (x + 1) = 5 \cdot 3 = 15.$$

Example: The function

$$f(x) = 2x \sin \sqrt{x}$$

satisfies the inequality

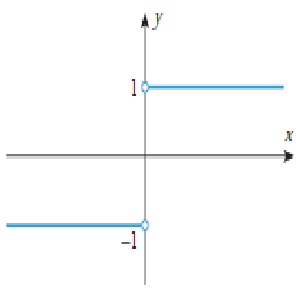
$$|f(x)| < \varepsilon, \qquad \text{whenever} \quad 0 < x < \delta = \varepsilon/2.$$

However, this does not mean that $\lim_{x\to 0} f(x) = 0$, since f is not defined for negative x.

Example: The function
$$g(x) = x + \frac{|x|}{x}, \quad x \neq 0,$$
can be rewritten as
$$g(x) = \begin{cases} x+1, & x > 0, \\ x-1, & x < 0; \end{cases}$$
hence, every open interval containing $x_0 = 0$ also contains points x_1 and x_2 such that $|g(x_1) - g(x_2)|$ is as close to 2 as we please. Therefore, $\lim_{x\to x_0} g(x)$ does not exist.

Although $f(x)$ and $g(x)$ do not approach limits as x approaches zero, they each exhibit a definite sort of limiting behavior for small positive values of x, as does $g(x)$ for small negative values of x. The kind of behavior we have in mind is defined precisely as follows.

Example: Investigate the limit
$$\lim_{x\to 0} \frac{|x|}{x}.$$



3.2 One Sided Limits

We say that $f(x)$ *approaches the left-hand limit* L *as* x *approaches* x_0 *from the left*, and write
$$\lim_{x\to x_0-} f(x) = L,$$
if f is defined on some open interval (a, x_0) and, for each $\varepsilon > 0$, there is a $\delta > 0$ such that
$$|f(x) - L| < \varepsilon \quad \text{if} \quad x_0 - \delta < x < x_0.$$
We say that $f(x)$ *approaches the right-hand limit* L *as* x *approaches* x_0 *from the right*, and write
$$\lim_{x\to x_0+} f(x) = L,$$

if f is defined on some open interval (x_0, b) and, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad x_0 < x < x_0 + \delta.$$

Example: Let

$$g(x) = \frac{x + |x|(1+x)}{x} \sin \frac{1}{x}, \quad x \neq 0.$$

Solution: If $x < 0$, then

$$g(x) = -x \sin \frac{1}{x}, \qquad \lim_{x \to 0-} g(x) = 0,$$

since

$$|g(x) - 0| = \left| x \sin \frac{1}{x} \right| \leq |x| < \varepsilon$$

if $-\varepsilon < x < 0$; that is, we have $\delta = \varepsilon$.

If $x > 0$, then

$$g(x) = (2 + x) \sin \frac{1}{x},$$

which takes on every value between -2 and 2 in every interval $(0, \delta)$.

Hence, $g(x)$ does not approach a right-hand limit at x approaches 0 from the right. This shows that a function may have a limit from one side at a point but fail to have a limit from the other side.

Example: Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution: See Lecture.

Example: Show that

$$\lim_{x \to 0+} \left(\frac{|x|}{x} + x \right) = 1, \qquad \lim_{x \to 0-} \left(\frac{|x|}{x} + x \right) = -1,$$

Solution: See Lecture

Theorem: A function f has a limit at x_0 if and only if it has left- and right-hand limits at x_0, and they are equal. More specifically,

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$\lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = L.$$

3.3 Limits at $\pm\infty$

If f is defined on an interval (a, ∞), then $f(x)$ *approaches the limit* L *as* x *approaches* ∞, and write

$$\lim_{x\to\infty} f(x) = L.$$

For each $\varepsilon > 0$, there is a number β such that

$$|f(x) - L| < \varepsilon \qquad \text{if} \quad x > \beta.$$

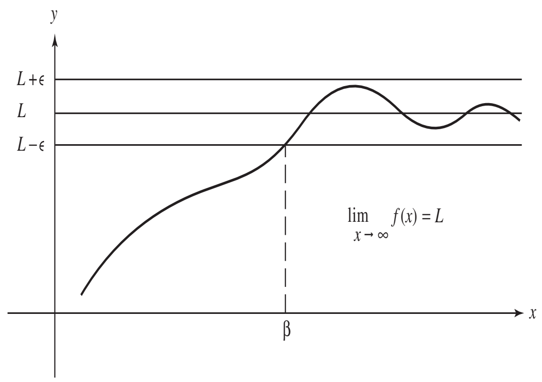


Figure 3.7: Limits at $\pm\infty$

Examples: Let

$$f(x) = 1 - \frac{1}{x^2}, \quad g(x) = \frac{2|x|}{1+x}, \quad \text{and} \quad h(x) = \sin x.$$

Find limits when $x \to \pm\infty$.

Solution: We have

$$\lim_{x\to\infty} f(x) = 1,$$

since

$$|f(x) - 1| = \frac{1}{x^2} < \varepsilon \quad \text{if} \quad x > \frac{1}{\sqrt{\varepsilon}}.$$

We have

$$\lim_{x\to\infty} g(x) = 2,$$

since

$$|g(x) - 2| = \left|\frac{2x}{1+x} - 2\right| = \frac{2}{1+x} < \frac{2}{x} < \varepsilon \quad \text{if} \quad x > \frac{2}{\varepsilon}.$$

However, $\lim_{x\to\infty} h(x)$ does not exist, since h assumes all values between -1 and 1 in any semi-infinite interval (τ, ∞). Discuss the limits at $x \to -\infty$.

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2},$$

and

$$p(x) = \sin\frac{1}{x},$$

$$q(x) = \frac{1}{x^2}\sin\frac{1}{x}$$

3.4 Infinite Limits

Infinite limits: We say that $f(x)$ *approaches* ∞ *as* x *approaches* x_0 *from the left*, and write

$$\lim_{x\to x_0-} f(x) = \infty.$$

If f is defined on an interval (a, x_0) and, for each real number M, there is a $\delta > 0$ such that

$$f(x) > M \quad \text{if} \quad x_0 - \delta < x < x_0.$$

Similarly, we can define the following limits

$$\lim_{x\to x_0-} f(x) = -\infty, \quad \lim_{x\to x_0+} f(x) = \infty, \quad \lim_{x\to x_0+} f(x) = -\infty.$$

$$\begin{aligned}
\lim_{x\to 0-}\frac{1}{x} &= -\infty, \\
\lim_{x\to 0+}\frac{1}{x} &= \infty; \\
\lim_{x\to 0-}\frac{1}{x^2} &= \lim_{x\to 0+}\frac{1}{x^2} = \infty \\
\lim_{x\to 0}\frac{1}{x^2} &= \infty; \\
\lim_{x\to\infty} x^2 &= \lim_{x\to-\infty} x^2 = \infty;
\end{aligned}$$

Example: If

$$f(x) = e^{2x} - e^x.$$

we cannot obtain $\lim_{x\to\infty} f(x)$ by writing

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{2x} - \lim_{x\to\infty} e^x,$$

because this produces the indeterminate form $\infty - \infty$.

However, by writing

$$f(x) = e^{2x}(1 - e^{-x}),$$

we find that

$$\lim_{x\to\infty} f(x) = \left(\lim_{x\to\infty} e^{2x}\right)\left(\lim_{x\to\infty} 1 - \lim_{x\to\infty} e^{-x}\right) = \infty(1-0) = \infty.$$

Example: Find

$$\lim_{x\to\infty} \frac{2x^2 - x + 1}{3x^2 + 2x - 1}.$$

Example: Find

$$\lim_{x\to 1} \frac{x^8 - 1}{x^4 - 1}.$$

$$f(x) = \begin{cases} \frac{1}{x+2}, & x < -2, \\ x^2 - 5, & -2 < x \le 3, \\ \sqrt{x + 13}, & x > 3. \end{cases}$$

3.5 Continuity

Continuity (Informal):

- We say that f is *continuous at* x_0 if f is defined on an open interval (a, b) containing x_0 and

$$\lim_{x\to x_0} f(x) = f(x_0).$$

- We say that f is *continuous from the left at* x_0 if f is defined on an open interval (a, x_0) and

$$\lim_{x\to x_0-} f(x) = f(x_0).$$

- We say that f is *continuous from the right at* x_0 if f is defined on an open interval (x_0, b) and

$$\lim_{x\to x_0+} f(x) = f(x_0).$$

Continuity (Formal):

- A function f is continuous at x_0 if and only if f is defined on an open interval (a, b) containing x_0 and for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

- A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0, b)$ and for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad x_0 \le x < x_0 + \delta.$$

- A function f is continuous from the left at x_0 if and only if f is defined on an interval $(a, x_0]$ and for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad x_0 - \delta < x \leq x_0.$$

Example: Discuss the continuity of f defined on $[0, 2]$ by

$$f(x) = \begin{cases} x^2, & 0 \leq x < 1, \\ x + 1, & 1 \leq x \leq 2 \end{cases}$$

Solution:

$$\begin{aligned} \lim_{x \to 0+} f(x) &= 0 = f(0), \\ \lim_{x \to 1-} f(x) &= 1 \neq f(1) = 2, \\ \lim_{x \to 1+} f(x) &= 2 = f(1), \\ \lim_{x \to 2-} f(x) &= 3 = f(2). \end{aligned}$$

Therefore, f is continuous from the right at 0 and 1 and continuous from the left at 2, but not at 1.

Example: Discuss the continuity of f defined on $[0, 2]$ by

$$f(x) = \begin{cases} x^2, & 0 \leq x < 1, \\ x + 1, & 1 \leq x \leq 2 \end{cases}$$

Solution: If $0 < x$, $x_0 < 1$, then

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| = |x - x_0|\,|x + x_0| \\ &\leq 2|x - x_0| < \varepsilon \quad \text{if} \quad |x - x_0| < \varepsilon/2. \end{aligned}$$

Hence, f is continuous at each x_0 in $(0, 1)$. If $1 < x$, $x_0 < 2$, then

$$\begin{aligned} |f(x) - f(x_0)| &= |(x + 1) - (x_0 + 1) = |x - x_0| \\ &< \varepsilon \quad \text{if} \quad |x - x_0| < \varepsilon. \end{aligned}$$

Hence, f is continuous at each x_0 in $(1, 2)$.

A function f is *continuous on an open interval* (a, b) if it is continuous at every point in (a, b).

If, in addition,

$$\lim_{x \to b_-} f(x) = f(b)$$

and

$$\lim_{x\to a_+} = f(a)$$

Example: Discuss the continuity of $f(x) = \sqrt{x}, \quad 0 \le x < \infty$.

Solution: Consider

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon \quad \text{if} \quad 0 \le x < \varepsilon^2,$$

so $\lim_{x\to 0_+} f(x) = f(0)$.

If $x_0 > 0$ and $x \ge 0$, then

$$\begin{aligned}|f(x) - f(x_0)| &= |\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \\ &\le \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon \quad \text{if} \quad |x - x_0| < \varepsilon\sqrt{x_0},\end{aligned}$$

so $\lim_{x\to x_0} f(x) = f(x_0)$. Hence, f is continuous on $[0, \infty)$.

Example: Discuss the continuity of $g(x) = \frac{1}{\sin \pi x}$

Solution: The function is continuous on

$$S = \bigcup_{n=-\infty}^{\infty} (n, n+1).$$

The function g is discontinuous on at any $x_0 = n$ (integer), since it is not defined at such points.

3.6 Piecewise Continuous Functions

Piecewise continuous function: A function f is *piecewise continuous* on $[a, b]$ if

- $\lim_{x\to x_{0_+}} f(x)$ exists for all x_0 in $[a, b)$;
- $\lim_{x\to x_{0_-}} f(x)$ exists for all x_0 in $(a, b]$;
- $\lim_{x\to x_{0_+}} f(x) = \lim_{x\to x_{0_-}} f(x) = f(x_0)$ for all but finitely many points x_0 in (a, b).

Example: Discuss the following function

$$f(x) = \begin{cases} 1, & x = 0, \\ x, & 0 < x < 1, \\ 2, & x = 1, \\ x, & 1 < x \leq 2, \\ -1, & 2 < x < 3, \\ 0, & x = 3. \end{cases}$$

Solution See lecture for explanation.

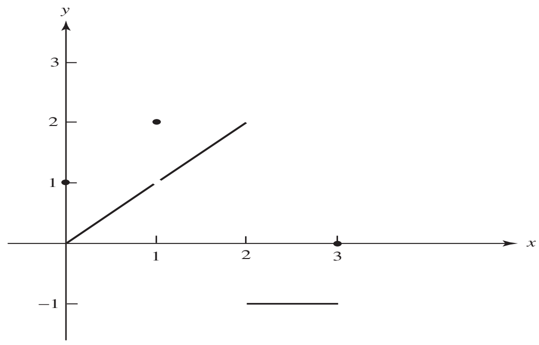


Figure 3.8: Graph of the function

Example: Discuss the following function

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0, \\ \\ 0, & x = 0, \end{cases}$$

Solution: The function is continuous at all x_0 except $x_0 = 0$.

As x approaches 0 from either side, $f(x)$ oscillates between -1 and 1 with ever-increasing frequency, so neither $\lim_{x \to 0_+} f(x)$ nor $\lim_{x \to 0_-} f(x)$ exists.

Therefore, the discontinuity of f at 0 is not a jump discontinuity, and if $\rho > 0$, then f is not piecewise continuous on any interval of the form $[-\rho, 0]$, $[-\rho, \rho]$, or $[0, \rho]$.

Theorem: If f and g are continuous on a set S, then so are $f + g$, $f - g$, and fg. In addition, f/g is continuous at each x_0 in S such that $g(x_0) \neq 0$.

Proof: See Lecture.

Example: Discuss the continuity of the following function

$$r(x) = \frac{9 - x^2}{x + 1}.$$

Solution: See Lecture.

3.7 Removable Discontinuity

It can be shown that if f_1, f_2, $\ldots$, f_n are continuous on a set S, then so are $f_1 + f_2 + \cdots + f_n$ and $f_1 f_2 \cdots f_n$.

Therefore, any rational function

$$r(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}$$

is continuous for all values of x except those for which its denominator vanishes.

Removable discontinuity: Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x \to x_0} f(x)$ exists.

In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

Example: Consider the function

$$f(x) = x \sin \frac{1}{x}.$$

Solution: The function is not defined at $x_0 = 0$, and therefore certainly not continuous there, but $\lim_{x \to 0} f(x) = 0$.

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

The function

$$f_1(x) = \sin \frac{1}{x}$$

is undefined at 0 and its discontinuity there is not removable, since $\lim_{x \to 0} f_1(x)$ does not exist.

Composition of functions: Suppose that f and g are functions with domains D_f and D_g.

If D_g has a nonempty subset T such that $g(x) \in D_f$ whenever $x \in T$, then the *composite function* $f \circ g$ is defined on T by

$$(f \circ g)(x) = f(g(x)).$$

Example: If

$$f(x) = \log x \quad \text{and} \quad g(x) = \frac{1}{1-x^2},$$

Can we define $f \circ g$?

Solution: The domain of the given functions are

$$D_f = (0, \infty) \quad \text{and} \quad D_g = x \in \mathbb{R}/\ x \neq \pm 1.$$

Since $g(x) > 0$ if $x \in T = (-1, 1)$, the composite function $f \circ g$ is defined on $(-1, 1)$ by

$$(f \circ g)(x) = \log \frac{1}{1-x^2}.$$

Can we define $g \circ f$?

The function $g \circ f$ is defined on $(0, 1/e) \cup (1/e, e) \cup (e, \infty)$ by

$$(g \circ f)(x) = \frac{1}{1-(\log x)^2}.$$

Theorem: Suppose that g is continuous at x_0, $g(x_0)$ is an interior point of D_f, and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.

Proof: Since $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$, for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $f(t)$ is defined and

$$|f(t) - f(g(x_0))| < \varepsilon \quad \text{if} \quad |t - g(x_0)| < \delta_1.$$

Since g is continuous at x_0, there is a $\delta > 0$ such that $g(x)$ is defined and

$$|g(x) - g(x_0)| < \delta_1 \quad \text{if} \quad |x - x_0| < \delta.$$

We can imply that

$$|f(g(x)) - f(g(x_0))| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta.$$

Therefore, $f \circ g$ is continuous at x_0.

Example: For the functions

$$f(x) = \sqrt{x}, \qquad g(x) = \frac{9-x^2}{x+1}.$$

Check the continuity of $f \circ g$.

Solution: The function f is continuous for $x > 0$, and the function g is continuous for $x \neq -1$.

Since $g(x) > 0$ if $x < -3$ or $-1 < x < 3$, the composite function

$$(f \circ g)(x) = \sqrt{\frac{9 - x^2}{x + 1}}$$

is continuous on $(-\infty, -3) \cup (-1, 3)$.

It is also continuous from the left at -3 and 3.

3.8 Bounded Functions

Bounded below function: A function f is *bounded below* on a set S if there is a real number m such that

$$f(x) \geq m \text{ for all } x \in S.$$

In this case, the set

$$V = \{f(x) : x \in S\}$$

has an infimum α, and we write

$$\alpha = \inf_{x \in S} f(x).$$

If there is a point x_1 in S such that $f(x_1) = \alpha$, we say that α is the *minimum of* f *on* S, and write

$$\alpha = \min_{x \in S} f(x).$$

Bounded above function: A function f is *bounded above on* S if there is a real number M such that $f(x) \leq M$ for all x in S.

In this case, V has a supremum β, and we write

$$\beta = \sup_{x \in S} f(x).$$

If there is a point x_2 in S such that $f(x_2) = \beta$, we say that β is the *maximum of* f *on* S, and write

$$\beta = \max_{x \in S} f(x).$$

If f is bounded above and below on a set S, we say that f is *bounded* on S.

Example: The function

$$g(x) = \begin{cases} \frac{1}{2}, & x = 0 \quad \text{or} \quad x = 1, \\ 1 - x, & 0 < x < 1, \end{cases} \qquad +$$

is bounded on $[0, 1]$, and

$$\sup_{0\le x\le 1} g(x) = 1, \quad \inf_{0\le x\le 1} g(x) = 0.$$

Therefore, g has no maximum or minimum on $[0, 1]$, since it does not assume either of the values 0 and 1.

Example: The function

$$h(x) = 1 - x, \quad 0 \le x \le 1,$$

which differs from g only at 0 and 1, has the same supremum and infimum as g, but it attains these values at $x = 0$ and $x = 1$, respectively.

Therefore,

$$\max_{0\le x\le 1} h(x) = 1 \quad \text{and} \quad \min_{0\le x\le 1} h(x) = 0.$$

Example: The function

$$f(x) = e^{x(x-1)} \sin \frac{1}{x(x-1)}, \quad 0 < x < 1,$$

oscillates between $\pm e^{x(x-1)}$ infinitely often in every interval of the form $(0, \rho)$ or $(1 - \rho, 1)$, where $0 < \rho < 1$.

Furthermore, we have

$$\sup_{0<x<1} f(x) = 1, \quad \inf_{0<x<1} f(x) = -1.$$

However, f does not assume these values, so f has no maximum or minimum on $(0, 1)$.

Theorem: If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$.

Proof: Suppose that $t \in [a, b]$. Since f is continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 \quad \text{if} \quad x \in I_t \cap [a, b].$$

The collection $\mathcal{H} = \{I_t : a \le t \le b\}$ is an open covering of $[a, b]$.

Since $[a, b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a, b]$.

According to continuity of the function and with $t = t_i$,

$$|f(x) - f(t_i)| < 1 \quad \text{if} \quad x \in I_{t_i} \cap [a, b].$$

Therefore,

$$\begin{aligned}|f(x)| &= |(f(x)-f(t_i))+f(t_i)| \le |f(x)-f(t_i)|+|f(t_i)| \\ &\le 1+|f(t_i)| \quad \text{if} \quad x \in I_{t_i}\cap[a,b].\end{aligned}$$

Let

$$M = 1 + \max_{1\le i\le n} |f(t_i)|.$$

Since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i}\cap[a,b])$, and we have $|f(x)| \le M$ if $x\in[a,b]$.

Theorem: Suppose that f is continuous on a finite closed interval $[a,b]$. Let

$$\alpha = \inf_{a\le x\le b} f(x) \quad \text{and} \quad \beta = \sup_{a\le x\le b} f(x).$$

Then α and β are respectively the minimum and maximum of f on $[a,b]$; that is, there are points x_1 and x_2 in $[a,b]$ such that

$$f(x_1)=\alpha \quad \text{and} \quad f(x_2)=\beta.$$

Proof: Suppose that there is no x_1 in $[a,b]$ such that $f(x_1)=\alpha$. Then $f(x)>\alpha$ for all $x\in[a,b]$. We will show that this leads to a contradiction.

Suppose that $t\in[a,b]$. Then $f(t)>\alpha$, so

$$f(t) > \frac{f(t)+\alpha}{2} > \alpha.$$

Since f is continuous at t, there is an open interval I_t about t such that

$$f(x) > \frac{f(t)+\alpha}{2} \quad \text{if} \quad x\in I_t\cap[a,b]. \tag{3.6}$$

The collection $\mathcal{H}=\{I_t : a\le t\le b\}$ is an open covering of $[a,b]$.

Since $[a,b]$ is compact, the Heine–Borel theorem implies that there are finitely many points t_1, t_2, …, t_n such that the intervals I_{t_1}, I_{t_2}, …, I_{t_n} cover $[a,b]$.

Define

$$\alpha_1 = \min_{1\le i\le n} \frac{f(t_i)+\alpha}{2}.$$

Then, since $[a,b]\subset\bigcup_{i=1}^{n}(I_{t_i}\cap[a,b])$, (3.6) implies that

$$f(t)>\alpha_1, \quad a\le t\le b.$$

But $\alpha_1>\alpha$, so this contradicts the definition of α. Therefore, $f(x_1)=\alpha$ for some x_1 in $[a,b]$.

Consider the function

$$g(x) = 1-(1-x)\sin\frac{1}{x}, \qquad (0,1].$$

The function $g(x)$ is continuous and has supremum 2 on the noncompact interval $(0,1]$.

Since

$$\begin{aligned} g(x) &\le 1 + (1-x)\left|\sin\frac{1}{x}\right| \\ &\le 1 + (1-x) < 2 \quad \text{if} \quad 0 < x \le 1. \end{aligned}$$

But does not assume its supremum on $(0,1]$.
Consider the function

$$f(x) = e^{-x}.$$

The function $f(x)$ is continuous and has infimum 0, which it does not attain, on the noncompact interval $(0,\infty)$.

3.9 The Intermediate Value Theorem

Theorem: Suppose that f is continuous on $[a,b]$, $f(a) \neq f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a,b).

Proof: Suppose that $f(a) < \mu < f(b)$. The set

$$S = x : a \le x \le b \quad \text{and} \quad f(x) \le \mu$$

is bounded and nonempty.

Let $c = \sup S$. We will show that $f(c) = \mu$.

If $f(c) > \mu$, then $c > a$ and, since f is continuous at c, there is an $\varepsilon > 0$ such that $f(x) > \mu$ if $c - \varepsilon < x \le c$.

Therefore, $c - \varepsilon$ is an upper bound for S, which contradicts the definition of c as the supremum of S.
If $f(c) < \mu$, then $c < b$ and there is an $\varepsilon > 0$ such that $f(x) < \mu$ for $c \le x < c+\varepsilon$, so c is not an upper bound for S.
This is also a contradiction. Therefore, $f(c) = \mu$.

3.10 Uniform Continuity

A function f is continuous on a subset S of its domain if for each $\varepsilon > 0$ and each x_0 in S, there is a $\delta > 0$, *which may depend upon* x_0 *as well as* ε, such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta \quad \text{and} \quad x \in D_f.$$

A function f is *uniformly continuous* on a subset S of its domain if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \text{ whenever } |x - x'| < \delta \text{ and } x, x' \in S.$$

Example: Check the uniform continuity of the function

$$f(x) = 2x.$$

Solution: For the function $f(x)$, we have

$$|f(x) - f(x')| = 2|x - x'| < \varepsilon \quad \text{if} \quad |x - x'| < \varepsilon/2.$$

The function $f(x)$ is uniformly continuous on $(-\infty, \infty)$,

Remark: A function f is *not* uniformly continuous on S if there is an $\varepsilon_0 > 0$ such that if δ is any positive number, there are points x and x' in S such that

$$|x - x'| < \delta$$

but

$$|f(x) - f(x')| \geq \varepsilon_0.$$

Example: Show that the function $g(x) = x^2$ is uniformly continuous on $[-r, r]$ for any finite r.

Solution: To see this, note that

$$|g(x) - g(x') = |x^2 - (x')^2| = |x - x'|\,|x + x'| \leq 2r|x - x'|,$$

so

$$|g(x) - g(x')| < \varepsilon \quad \text{if} \quad |x - x'| < \delta = \frac{\varepsilon}{2r} \quad \text{and} \quad -r \leq x, x' \leq r.$$

Example: The function $g(x) = x^2$ is not uniformly continuous on $(-\infty, \infty)$.

Solution: To see this, we will show that if $\delta > 0$ there are real numbers x and x' such that

$$|x - x'| = \delta/2 \quad \text{and} \quad |g(x) - g(x')| \geq 1.$$

To this end, we write

$$|g(x) - g(x')| = |x^2 - (x')^2| = |x - x'|\,|x + x'|.$$

If $|x - x'| = \delta/2$ and $x, x' > 1/\delta$, then

$$|x - x'|\,|x + x'| > \frac{\delta}{2}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) = 1.$$

Example: The function
$$f(x) = \cos\frac{1}{x}$$
is continuous on $(0, 1]$

However, f is not uniformly continuous on $(0, 1]$, since
$$\left| f\left(\frac{1}{n\pi}\right) - f\left(\frac{1}{(n+1)\pi}\right) \right| = 2, \quad n = 1, 2, \ldots.$$

Theorem: If f is continuous on a closed and bounded interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof: Suppose that $\varepsilon > 0$. Since f is continuous on $[a, b]$, for each t in $[a, b]$ there is a positive number δ_t such that
$$|f(x) - f(t)| < \frac{\varepsilon}{2} \quad \text{if} \quad |x - t| < 2\delta_t \quad \text{and} \quad x \in [a, b].$$

If $I_t = (t - \delta_t, t + \delta_t)$, the collection
$$\mathcal{H} = \{I_t : t \in [a, b]\}$$
is an open covering of $[a, b]$.

Since $[a, b]$ is compact, the Heine–Borel theorem implies that there are finitely many points t_1, t_2, $\ldots$, t_n in $[a, b]$ such that I_{t_1}, I_{t_2}, $\ldots$, I_{t_n} cover $[a, b]$.

Now define
$$\delta = \min\{\delta_{t_1}, \delta_{t_2}, \ldots, \delta_{t_n}\}. \tag{3.7}$$
We will show that if
$$|x - x'| < \delta \quad \text{and} \quad x, x' \in [a, b], \tag{3.8}$$
then $|f(x) - f(x')| < \varepsilon$.

From the triangle inequality,
$$\begin{aligned} |f(x) - f(x')| &= |\,(f(x) - f(t_r)) + (f(t_r) - f(x'))\,| \\ &\leq |f(x) - f(t_r)| + |f(t_r) - f(x')|. \end{aligned} \tag{3.9}$$

Since I_{t_1}, I_{t_2}, $\ldots$, I_{t_n} cover $[a, b]$, x must be in one of these intervals. Suppose that $x \in I_{t_r}$; that is,
$$|x - t_r| < \delta_{t_r}. \tag{3.10}$$

From continuity condition with $t = t_r$,
$$|f(x) - f(t_r)| < \frac{\varepsilon}{2}. \tag{3.11}$$

From (3.8), (3.10), and the triangle inequality,

$$|x' - t_r| = |(x' - x) + (x - t_r)| \le |x' - x| + |x - t_r| < \delta + \delta_{t_r} \le 2\delta_{t_r}.$$

Therefore, from continuity condition with $t = t_r$ and x replaced by x' implies that

$$|f(x') - f(t_r)| < \frac{\varepsilon}{2}.$$

This, (3.9), and (3.11) imply that $|f(x) - f(x')| < \varepsilon$.

3.11 Monotonic Functions

Monotonic functions: A function f is *nondecreasing* on an interval I if

$$f(x_1) \le f(x_2) \quad \text{whenever} \quad x_1, x_2 \in I, \quad x_1 < x_2. \tag{3.12}$$

or *nonincreasing* on I if

$$f(x_1) \ge f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2. \tag{3.13}$$

In either case, f is on I.

If $\le$ can be replaced by $<$ in (3.12), f is *increasing* on I. If $\ge$ can be replaced by $>$ in (3.13), f is *decreasing* on I. In either of these two cases, f is *strictly monotonic* on I.

Example: The function

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ 2, & 1 \le x \le 2, \end{cases}$$

is nondecreasing on $I = [0, 2]$.

Example: The function $g(x) = x^2$ is increasing on $[0, \infty)$. The function $h(x) = -x^3$ is decreasing on $(-\infty, \infty)$.

Theorem: Suppose that f is monotonic on (a, b) and define

$$\alpha = \inf_{a<x<b} f(x) \quad \text{and} \quad \beta = \sup_{a<x<b} f(x).$$

1. If f is nondecreasing, then $\lim_{x\to a_+} f(x) = \alpha$ and $\lim_{x\to b_-} f(x) = \beta$.
2. If f is nonincreasing, then $\lim_{x\to a_+} f(x) = \beta$ and $\lim_{x\to b_-} f(x) = \alpha$. (Here $a+ = -\infty$ if $a = -\infty$ and $b- = \infty$ if $b = \infty$.)

3. If $a < x_0 < b$, then $\lim_{x\to x_{0+}} f(x)$ and $\lim_{x\to x_{0-}} f(x)$ exist and are finite; moreover,
$$\lim_{x\to x_{0-}} f(x) \le f(x_0) \le \lim_{x\to x_{0+}} f(x)$$
if f is nondecreasing, and
$$\lim_{x\to x_{0-}} f(x) \ge f(x_0) \ge \lim_{x\to x_{0+}} f(x)$$
if f is nonincreasing.

Proof: We first show that $\lim_{x\to a_+} f(x) = \alpha$.

If $M > \alpha$, there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty$, then $\lim_{x\to a_+} f(x) = -\infty$.

If $\alpha > -\infty$, let $M = \alpha + \varepsilon$, where $\varepsilon > 0$.

Then $\alpha \le f(x) < \alpha + \varepsilon$, so
$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < x_0. \tag{3.14}$$

If $a = -\infty$, this implies that $f(-\infty) = \alpha$. If $a > -\infty$, let $\delta = x_0 - a$. Then (3.14) is equivalent to
$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < a + \delta,$$
which implies that $f(a+) = \alpha$.

We now show that $\lim_{x\to b_-} f(x) = \beta$.

If $M < \beta$, there is an x_0 in (a, b) such that $f(x_0) > M$.

Since f is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty$, then $\lim_{x\to b_-} f(x) = \infty$.

If $\beta < \infty$, let $M = \beta - \varepsilon$, where $\varepsilon > 0$. Then $\beta - \varepsilon < f(x) \le \beta$, so
$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad x_0 < x < b. \tag{3.15}$$

If $b = \infty$, this implies that $f(\infty) = \beta$. If $b < \infty$, let $\delta = b - x_0$.

Then (3.15) is equivalent to
$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad b - \delta < x < b,$$
which implies that $f(b-) = \beta$.

3.12 Limits Inferior and Superior

Suppose that f is bounded on $[a, x_0)$, where x_0 may be finite or ∞.

For $a \le x < x_0$, define
$$\begin{aligned} S_f(x; x_0) &= \sup_{x\le t<x_0} f(t) \\ I_f(x; x_0) &= \inf_{x\le t<x_0} f(t). \end{aligned}$$

Then the *left limit superior of f at x_0* is defined to be

$$\limsup_{x\to x_{0-}} f(x) = \lim_{x\to x_{0-}} S_f(x;x_0).$$

The *left limit inferior of f at x_0* is defined to be

$$\liminf_{x\to x_{0-}} f(x) = \lim_{x\to x_{0-}} I_f(x;x_0).$$

(If $x_0=\infty$, we define $x_0-=\infty$.)

Theorem: If f is bounded on $[a,x_0)$, then $\beta=\limsup_{x\to x_0-} f(x)$ exists and is the unique real number with the following properties:

1. If $\varepsilon>0$, there is an a_1 in $[a,x_0)$ such that

$$f(x)<\beta+\varepsilon \quad \text{if} \quad a_1\le x<x_0.$$

2. If $\varepsilon>0$ and a_1 is in $[a,x_0)$, then

$$f(\overline{x})>\beta-\varepsilon \quad \text{for some } \overline{x}\in[a_1,x_0).$$

Theorem: If f is bounded on $[a,x_0)$, then $\alpha=\liminf_{x\to x_{0-}} f(x)$ exists and is the unique real number with the following properties:

1. If $\varepsilon>0$, there is an a_1 in $[a,x_0)$ such that

$$f(x)>\alpha-\varepsilon \quad \text{if} \quad a_1\le x<x_0.$$

2. If $\varepsilon>0$ and a_1 is in $[a,x_0)$, then

$$f(\overline{x})<\alpha+\varepsilon \quad \text{for some } \overline{x}\in[a_1,x_0).$$

Theorem: Suppose that f is monotonic on (a,b) and define

$$\alpha=\inf_{a<x<b} f(x), \quad \beta=\sup_{a<x<b} f(x).$$

1. If f is nondecreasing, then $\lim_{x\to a_+} f(x)=\alpha$ and $\lim_{x\to b_-} f(x)=\beta$.

2. If $a<x_0<b$, then $\lim_{x\to x_{0+}} f(x)$ and $\lim_{x\to x_{0-}} f(x)$ exist and are finite; moreover,

$$\lim_{x\to x_{0-}} f(x)\le f(x_0)\le \lim_{x\to x_{0+}} f(x).$$

Theorem: If f is monotonic and nonconstant on $[a,b]$, then f is continuous on $[a,b]$ if and only if its range $R_f = \{f(x) : x \in [a,b]\}$ is the closed interval with endpoints $f(a)$ and $f(b)$.

Proof: We assume that f is nondecreasing. The theorem implies that the set $\widetilde{R}_f = \{f(x) : x \in (a,b)\}$ is a subset of the open interval $(\lim_{x\to a_+} f(x), \lim_{x\to b_-} f(x))$.

Therefore,

$$R_f = \{f(a)\} \cup \widetilde{R}_f \cup \{f(b)\} \subset \{f(a)\} \cup (\lim_{x\to a_+} f(x), \lim_{x\to b_-} f(x)) \cup \{f(b)\}. \quad (3.16)$$

Now suppose that f is continuous on $[a,b]$. Then $f(a) = \lim_{x\to a_+} f(x)$, $\lim_{x\to b_-} f(x) = f(b)$. So (3.16) implies that $R_f \subset [f(a), f(b)]$.

If $f(a) < \mu < f(b)$, then by Intermediate Value Theorem implies that $\mu = f(x)$ for some x in (a,b). Hence, $R_f = [f(a), f(b)]$.

For the converse, suppose that $R_f = [f(a), f(b)]$. Since $f(a) \le \lim_{x\to a_+} f(x)$ and $\lim_{x\to b_-} f(x) \le f(b)$, (3.16) implies that $f(a) = \lim_{x\to a_+} f(x)$ and $\lim_{x\to b_-} f(x) = f(b)$.

We know that if f is nondecreasing and $a < x_0 < b$, then

$$\lim_{x\to x_0-} f(x) \le f(x_0) \le \lim_{x\to 0_+} f(x).$$

If either of these inequalities is strict, R_f cannot be an interval. Since this contradicts our assumption, $\lim_{x\to x_{0_-}} f(x) = f(x_0) = \lim_{x\to x_{0_+}} f(x)$.

Therefore, f is continuous at x_0. We can now conclude that f is continuous on $[a,b]$.

Theorem: Suppose that f is increasing and continuous on $[a,b]$, and let $f(a) = c$ and $f(b) = d$.

Then there is a unique function g defined on $[c,d]$ such that

$$g(f(x)) = x, \quad a \le x \le b,$$

and

$$f(g(y)) = y, \quad c \le y \le d.$$

Moreover, g is continuous and increasing on $[c,d]$.

Proof: **Step I** There is a function g satisfying the above two equations.

Step II: Uniqueness of the function g.

Step II: g is increasing and continuous.

Since f is continuous, then for each y_0 in $[c,d]$ there is an x_0 in $[a,b]$ such that

$$f(x_0) = y_0,$$

and, since f is increasing, there is only one such x_0.

Define

$$g(y_0) = x_0, \tag{3.17}$$

we have

$$f(g(y_0)) = y_0, \qquad g(f(x_0)) = x_0.$$

Since this is true for all x_0 and y_0, so

$$g(f(x)) = x, \quad a \le x \le b, \quad f(g(y)) = y, \quad c \le y \le d.$$

The uniqueness of g follows from our assumption that f is increasing, and therefore only one value of x_0 can satisfy $f(x_0) = y_0$, for each y_0.

To see that g is increasing, suppose that $y_1 < y_2$ and let x_1 and x_2 be the points in $[a, b]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Since f is increasing, $x_1 < x_2$. Therefore,

$$g(y_1) = x_1 < x_2 = g(y_2),$$

so g is increasing.

Since $R_g = \{g(y) : y \in [c, d]\}$ is the interval $[g(c), g(d)] = [a, b]$, therefore from theorem, we have proved with f and $[a, b]$ replaced by g and $[c, d]$ implies that g is continuous on $[c, d]$.

Example: If

$$f(x) = x^2, \quad 0 \le x \le R.$$

Solution: The inverse of the given function is

$$f^{-1}(y) = g(y) = \sqrt{y}, \quad 0 \le y \le R^2.$$

Example: If

$$f(x) = 2x + 4, \quad 0 \le x \le 2,$$

Solution: The inverse of the given function is

$$f^{-1}(y) = g(y) = \frac{y-4}{2}, \quad 4 \le y \le 8.$$

Chapter 4

Differentiability

4.1 Derivative

Derivative: A function f is *differentiable* at an interior point x_0 of its domain if the difference quotient

$$\frac{f(x)-f(x_0)}{x-x_0}, \quad x \neq x_0,$$

approaches a limit as x approaches x_0, in which case the limit is called the *derivative of f at x_0*, and is denoted by $f'(x_0)$.

Thus,

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

If we take $x = x_0 + h$ then

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

- If f is defined on an open set S, we say that f is *differentiable on S* if f is differentiable at every point of S.
- If f is differentiable on S, then f' is a function on S. We say that f is *continuously differentiable* on S if f' is continuous on S.
- If f is differentiable on a neighborhood of x_0, it is reasonable to ask if f' is differentiable at x_0. If so, we denote the derivative of f' at x_0 by $f''(x_0)$.
- This is the *second derivative of f at x_0*, and it is also denoted by $f^{(2)}(x_0)$. Continuing inductively, if $f^{(n-1)}$ is defined on a neighborhood of x_0, then the *nth derivative of f at x_0*, denoted by $f^{(n)}(x_0)$, is the derivative of $f^{(n-1)}$ at x_0.
- This is the *second derivative of f at x_0*, and it is also denoted by $f^{(2)}(x_0)$.
- Continuing inductively, if $f^{(n-1)}$ is defined on a neighborhood of x_0, then the *nth derivative of f at x_0*, denoted by $f^{(n)}(x_0)$, is the derivative of $f^{(n-1)}$ at x_0.
- For convenience we define the *zeroth derivative* of f to be f itself; thus

$$f^{(0)}=f.$$

We assume that you are familiar with the other standard notations for derivatives; for example,

$$f^{(2)} = f'', \quad f^{(3)} = f''',$$
$$\frac{d^n f}{dx^n} = f^{(n)}.$$

Example: If n is a positive integer, find the derivative of the function

$$f(x) = x^n.$$

Solution: We have

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^n - x_0^n}{x - x_0} = \frac{x - x_0}{x - x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k,$$

$$f'(x_0) = \lim_{x \to x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k = n x_0^{n-1}.$$

Since this holds for every x_0, we drop the subscript and write

$$f'(x) = nx^{n-1} \quad \text{or} \quad \frac{d}{dx}(x^n) = nx^{n-1}.$$

Example: Find the derivative of the line $y = mx + c$.

Solution:

$$\begin{aligned} &\frac{f(x) - f(x_0)}{x - x_0} \\ &= \frac{mx + c - (mx_0 + c)}{x - x_0} \\ &= \frac{m(x - x_0)}{x - x_0} = m \end{aligned}$$

Consequently

$$f'(x_0) = \lim_{x \to x_0} m = m.$$

Geometrical interpretation of derivative: The equation of the line through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the curve $y = f(x)$ is

$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

Varying x_1 generates lines through $(x_0, f(x_0))$ that rotate into the line

$$y = f(x_0) + f'(x_0)(x - x_0)$$

as x_1 approaches x_0. This is the *tangent* to the curve $y = f(x)$ at the point $(x_0, f(x_0))$.

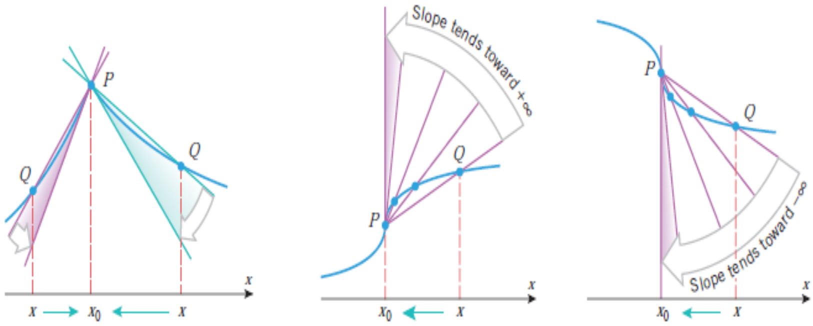


Figure 4.1: The tangent lines

Lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

Proof: Define

$$E(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0), & x \in D_f \text{ and } x \neq x_0, \\ 0, & x = x_0. \end{cases}$$

Apply that lemma to $f(x) = x^2$ at $x_0 = 3$.

4.1.1 Differentiability Implies Continuity

Theorem: If f is differentiable at x_0, then f is continuous at x_0.

Proof:

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

where

$$E(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0), & x \in D_f \text{ and } x \neq x_0, \\ 0, & x = x_0. \end{cases}$$

Examples:

- $f(x) = x^2$ has derivative $f'(x) = 4x$.
- $f(x) = \frac{1}{x}$ has derivative $f'(x) = \frac{-1}{x^2}$.
- $f(x) = \sin x$ has derivative $f'(x) = \cos x$.

Is continuity implies differentiability?

Counter example: Consider the function

$$f(x) = |x|.$$

The functions can be written as

$$f(x) = \begin{cases} x, & x > 0, \\ -x, & x \leq 0, \end{cases} \qquad \Rightarrow \qquad f'(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0, \end{cases}$$

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x}{x} \tag{4.1}$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1 \tag{4.2}$$

are different,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist (Theorem **??**); thus, f is not differentiable at 0, even though it is continuous at 0.

Theorem: If f and g are differentiable at x_0, then so are $f+g$, $f-g$, and fg, with

1. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;
2. $(f-g)'(x_0) = f'(x_0) - g(x_0)$;
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

The quotient f/g is differentiable at x_0 if $g(x_0) \neq 0$, with

- $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$.

Proof: The trick is to add and subtract the right quantity in the numerator of the difference quotient for $(fg)'(x_0)$; thus,

$$\begin{aligned} & \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ = & \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ = & \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}. \end{aligned}$$

The difference quotients on the right approach $f'(x_0)$ and $g'(x_0)$ as x approaches x_0, and $\lim_{x \to x_0} g(x) = g(x_0)$.

Example: Find $\frac{ds}{dt}$ where
$s(t) = (5 - t^2)t^{3/2}$.

Example: Find $\frac{dy}{dx}$ where
$y(x) = \frac{x^4 - 5x^2}{x - 5}$.

Lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

Theorem (The chain rule): Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$.

Then the composite function $h = f \circ g$, defined by

$$h(x) = f(g(x)),$$

is differentiable at x_0, with

$$h'(x_0) = f'(g(x_0))g'(x_0).$$

Proof: Since f is differentiable at $g(x_0)$, we can write

$$f(t) - f(g(x_0)) = [f'(g(x_0)) + E(t)][t - g(x_0)],$$

where

$$\lim_{t \to g(x_0)} E(t) = E(g(x_0)) = 0. \tag{4.3}$$

Letting $t = g(x)$ yields

$$f(g(x)) - f(g(x_0)) = [f'(g(x_0)) + E(g(x))][g(x) - g(x_0)].$$

Since $h(x) = f(g(x))$, this implies that

$$\frac{h(x) - h(x_0)}{x - x_0} = [f'(g(x_0)) + E(g(x))]\frac{g(x) - g(x_0)}{x - x_0}. \tag{4.4}$$

Since g is continuous at x_0 we have

$$\lim_{x \to x_0} E(g(x)) = E(g(x_0)) = 0.$$

Therefore, (4.4) implies that

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = f'(g(x_0))g'(x_0),$$

as stated.

Example: Calculate the derivative of

$$h(x) = \sin \frac{1}{x}, \quad x \neq 0.$$

If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x)) = \sin \frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$

Example: What is wrong in the following justification?

$$\begin{aligned} \frac{h(x) - h(x_0)}{x - x_0} &= \frac{f(g(x)) - f(g(x_0))}{x - x_0} \\ &= \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \, \frac{g(x) - g(x_0)}{x - x_0} \end{aligned}$$

and arguing that

$$\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0))$$

(because $\lim_{x \to x_0} g(x) = g(x_0)$) and

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0).$$

Is it a valid proof?

4.2 One Sided Derivative

One sided derivatives: If f is defined on $[x_0, b)$, the *right-hand derivative of f at x_0* is defined to be

$$f'_+(x_0) = \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0}.$$

if the limit exists.

If f is defined on $(a, x_0]$, the *left-hand derivative of f at x_0* is defined to be

$$f'_-(x_0) = \lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$$

if the limit exists.

The definition of limit implies that f is differentiable at x_0 if and only if $f'_+(x_0)$ and $f'_-(x_0)$ exist and are equal, in which case

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

The functions defined by $f(x) = |x|$ and $f(x) = \frac{|x|}{x}$ have one sided derivatives.

Example: For the piecewise defined function

$$f(x) = \begin{cases} x^3, & x \le 0, \\ x^2 \sin\frac{1}{x}, & x > 0, \end{cases}$$

Investigate the one sided derivatives.

Solution: For the given function, we have

$$f'(x) = \begin{cases} 3x^2, & x < 0, \\ 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x > 0. \end{cases}$$

Since neither formula in $f(x)$ holds for all x in any neighborhood of 0, we cannot simply differentiate either to obtain $f'(0)$.

We will calculate the one sided derivatives

$$\begin{aligned} \lim_{x\to 0+} f(x) &= \lim_{x\to 0+} \frac{x^2\sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0+} x\sin\frac{1}{x} = 0, \\ f'_-(0) &= \lim_{x\to 0-} \frac{x^3 - 0}{x - 0} = \lim_{x\to 0-} x^2 = 0; \end{aligned}$$

hence, $f'(0) = f'_+(0) = f'_-(0) = 0$.

Remark: There is a difference between one sided derivatives and one sided limit of derivative

4.3 Differentiable Function

Differentiable function: A function f is *differentiable on the closed interval* $[a, b]$ if f is differentiable on the open interval (a, b) and $f'_{+}(a)$ and $f'_{-}(b)$ both exist.

Continuously differentiable We say that f is *continuously differentiable on* $[a, b]$ if f is differentiable on $[a, b]$, f' is continuous on (a, b),

$$f'_{+}(a) = \lim_{x \to a+} f'(x)$$

, and

$$f'_{-}(b) = \lim_{x \to b_{-}} f'(x).$$

4.4 Extreme Values of a Function

We say that $f(x_0)$ is a *local extreme value* of f if there is a $\delta > 0$ such that $f(x) - f(x_0)$ does not change sign on

$$(x_0 - \delta, x_0 + \delta) \cap D_f.$$

More specifically, $f(x_0)$ is a *local maximum value* of f if

$$f(x) \leq f(x_0)$$

or a *local minimum value* of f if

$$f(x) \geq f(x_0)$$

for all x in the set $(x_0 - \delta, x_0 + \delta) \cap D_f$.

The point x_0 is called a *local extreme point* of f, or, more specifically, a *local maximum* or *local minimum point* of f.

Example: For the function

$$f(x) = \begin{cases} 1, & -1 < x \leq -\frac{1}{2} \\ |x|, & -\frac{1}{2} < x \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2}} \sin \frac{\pi x}{2}, & \frac{1}{2} < x \leq 4 \end{cases}$$

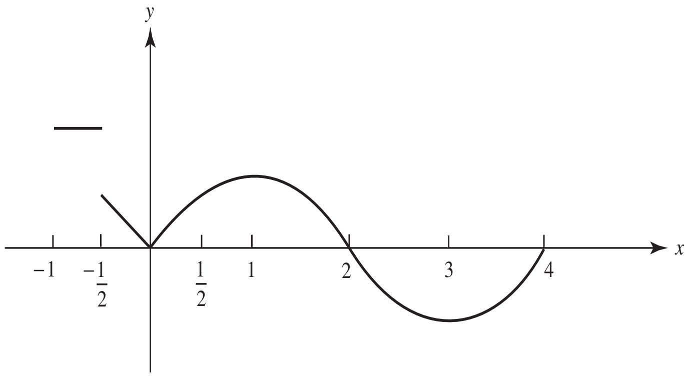


Figure 4.2: Extreme values of a function

Recall the Lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

Theorem: If f is differentiable at a local extreme point $x_0 \in D_f^0$, then $f'(x_0) = 0$.

Proof: We will show that x_0 is not a local extreme point of f if $f'(x_0) \neq 0$.

From Lemma, we have

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + E(x),$$

where $\lim_{x \to x_0} E(x) = 0$.

Therefore, if $f'(x_0) \neq 0$, there is a $\delta > 0$ such that

$$|E(x)| < |f'(x_0)| \quad \text{if} \quad |x - x_0| < \delta, \qquad (*)$$

and the right side of (*) must have the same sign as $f'(x_0)$ for $|x - x_0| < \delta$. Since the same is true of the left side, $f(x) - f(x_0)$ must change sign in every neighborhood of x_0 (since $x - x_0$ does).

- If $f'(x_0) = 0$ then x_0 is said to be a critical point.

Recall the following: If a function f is continuous on the closed interval then f attains its extreme values in the closed interval.

Theorem: If f is differentiable at a local extreme point x_0, then $f'(x_0) = 0$.

4.5 Rolle's Theorem

Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a, b).

Proof: Since f is continuous on $[a, b]$, f attains a maximum and a minimum value on $[a, b]$. If these two extreme values are the same, then f is constant on (a, b), so $f'(x) = 0$ for all x in (a, b).

If the extreme values differ, then at least one must be attained at some point c in the open interval (a, b), and $f'(c) = 0$.

4.6 The Mean Value Theorem

Theorem: Suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some c in (a, b).

Proof: Suppose first that

$$f'(a) < \mu < f'(b)$$

and define

$$g(x) = f(x) - \mu x.$$

Then

$$g'(x) = f'(x) - \mu, \quad a \leq x \leq b,$$

and by our supposition $f'(a) < \mu < f'(b)$, we have

$$g'(a) < 0 \quad \text{and} \quad g'(b) > 0.$$

Since g is continuous on $[a, b]$, g attains a minimum at some point c in $[a, b]$.

Due to Lemma and $g'(a) < 0 \quad$ and $\quad g'(b) > 0$, imply that there is a $\delta > 0$ such that

$$g(x) < g(a), \quad a < x < a + \delta, \quad \text{and} \quad g(x) < g(b), \quad b - \delta < x < b.$$

Therefore $c \neq a$ and $c \neq b$. Hence, $a < c < b$, and therefore $g'(c) = 0$, by the fact that if f is differentiable at a local extreme point x_0 then $f'(x_0) = 0$. From , $f'(c) = \mu$. The proof for the case where $f'(b) < \mu < f'(a)$ can be obtained by applying this result to $-f$.

Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some c in (a, b).

Proof: The function

$$h(x) = [b - a]f(x) - [f(b) - f(a)]x$$

is continuous on $[a, b]$ and differentiable on (a, b).

Furthermore

$$h(a) = h(b) = bf(a) - f(b)a.$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a, b).

$$\text{Since} \quad h'(c) = [b - a]f'(c) - [f(b) - f(a)].$$

4.7 Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for some c in (a, b).

Proof: The function

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$$

is continuous on $[a, b]$ and differentiable on (a, b).

Furthermore

$$h(a) = h(b) = g(b)f(a) - f(b)g(a).$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a, b).

$$\text{Since} \quad h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c).$$

Theorem: If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Proof: See Lecture.

Theorem: If f' exists and does not change sign on (a, b), then f is monotonic on (a, b) : increasing, nondecreasing, decreasing, or nonincreasing as

$$f'(x) > 0, \quad f'(x) \geq 0, \quad f'(x) < 0, \quad \text{or} \quad f'(x) \leq 0,$$

respectively, for all x in (a, b).

Proof: See Lecture.

Theorem: If f and g are differentiable on an interval, and if $f'(x) = g'(x)$ for all x in that interval, then $f - g$ is a constant function on the interval.

Proof: See Lecture.

4.8 Lipschitz Continuity

A function that satisfies inequality

$$|f(x) - f(x')| \le M|x - x'|, \quad x, x' \in (a, b),$$

for all x and x' in an interval, where $M > 0$ is some real number is said to satisfy a *Lipschitz condition* on the interval.

Remark: A differentiable real valued function is Lipschitz continuous, indeed we have

$$\text{differentiability at } x \Rightarrow \text{Lipschits continuity at } x \Rightarrow \text{Continuity at } x$$

Lipschitz continuity: Continuity at $x \not\Rightarrow$ Lipschits continuity at $x \not\Rightarrow$ differentiability at x

Example: The function $f(x) = \sqrt{|x|}$ is continuous at $x = 0$, but not Lipschits continuous at $x = 0$.

The function $f(x) = |x|$ is Lipschits continuous at $x = 0$ but not differentiable at $x = 0$.

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c)$$

$$= [f(b) - f(a)]g'(c)$$

for some c in (a, b).

Theorem: Suppose that f and g are differentiable and g' has no zeros on (a, b).

Let

$$\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0$$

or

$$\lim_{x \to b-} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to b-} g(x) = \pm\infty,$$

and suppose that

$$\lim_{x \to b-} \frac{f'(x)}{g'(x)} = L \quad (\text{finite or} \pm \infty).$$

Then

$$\lim_{x \to b-} \frac{f(x)}{g(x)} = L.$$

Proof: For $\varepsilon > 0$, due to definition of limit, there is an x_0 in (a, b) such that

$$\left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon \quad \text{if} \quad x_0 < c < b.$$

Generalized mean value theorem implies that if x and t are in $[x_0, b)$, then there is a c between them, and therefore in (x_0, b), such that

$$[g(x) - g(t)]f'(c) = [f(x) - f(t)]g'(c).$$

Since g' has no zeros in (a, b), mean value theorem implies that

$$g(x) - g(t) \neq 0 \quad \text{if} \quad x, t \in (a, b).$$

This means that g cannot have more than one zero in (a, b).

Therefore, we can choose x_0 so that, in addition to satisfy

$$\left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon \quad \text{if} \quad x_0 < c < b.$$

g has no zeros in $[x_0, b)$.

Then we can write

$$\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)},$$

so

$$\left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon \quad \text{if} \quad x_0 < c < b.$$

implies that

$$\left|\frac{f(x) - f(t)}{g(x) - g(t)} - L\right| < \varepsilon \quad \text{if} \quad x, t \in [x_0, b). \tag{4.5}$$

If $\lim_{x\to b-} f(x) = \lim_{x\to b-} g(x) = 0$ holds, let x be fixed in $[x_0, b)$, and consider the function

$$G(t) = \frac{f(x) - f(t)}{g(x) - g(t)} - L.$$

so

$$\lim_{t\to b-} G(t) = \frac{f(x)}{g(x)} - L.$$

Since

$$|G(t)| < \varepsilon \quad \text{if} \quad x_0 < t < b,$$

we have

$$\left|\frac{f(x)}{g(x)} - L\right| \leq \varepsilon.$$

This holds for all x in (x_0, b), which is the required result.

- For the proof $\lim_{x\to b-} f(x) = \pm\infty \lim_{x\to b-} g(x) = \pm\infty$

- and when $L = \pm\infty$, try yourself.

4.9 Indeterminate Forms

The quotient function f/g *is of the form* $0/0$ *as* $x \to b-$ if

$$\lim_{x\to b-} f(x) = \lim_{x\to b-} g(x) = 0,$$

or *of the form* ∞/∞ *as* $x \to b-$ if

$$\lim_{x\to b-} f(x) = \pm\infty$$

and

$$\lim_{x\to b-} g(x) = \pm\infty.$$

The corresponding definitions for $x \to b+$ and $x \to \pm\infty$ are similar.

If f/g is of one of these forms as $x \to b-$ and as $x \to b+$, then we say that it is of that form as $x \to b$.

Example: Evaluate the following limit

$$\lim_{x\to 0} \sin x/x.$$

Solution: The given limit limit $\lim_{x\to 0} \sin x/x$. is of the form $0/0$ as $x \to 0$, and L'Hospital's rule yields

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example: Evaluate the limit

$$\lim_{x\to -\infty} \frac{e^{-x}}{x}.$$

Solution: See Lecture.

Example: Evaluate the limit

$$\lim_{x\to +\infty} \frac{x^{-4/3}}{\sin(1/x)}.$$

Solution: See Lecture.

Example: Using L'Hospital's rule may lead to another indeterminate form; thus,

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

if the limit on the right exists in the extended reals. Applying L'Hospital's rule again yields

$$\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty.$$

Therefore,

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\infty.$$

More generally,

$$\lim_{x\to\infty}\frac{e^x}{x^\alpha}=\infty$$

for any real number α

Example: Evaluate $\lim_{x\to 0}\frac{4-4\cos x-2\sin^2 x}{x^4}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0}\frac{4-4\cos x-2\sin^2 x}{x^4} &= \lim_{x\to 0}\frac{4\sin x-4\sin x\cos x}{4x^3}\\
&= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1-\cos x}{x^2}\right)\\
&= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{2x}\right)\\
&= \frac{1}{2}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\\
&= \frac{1}{2}(1)^2=\frac{1}{2}.
\end{aligned}$$

Example: If

$$f(x)=x-x^2\sin\frac{1}{x}\quad\text{and}\quad g(x)=\sin x,$$

then

$$f'(x)=1-2x\sin\frac{1}{x}+\cos\frac{1}{x}\quad\text{and}\quad g'(x)=\cos x.$$

Therefore, $\lim_{x\to 0}f'(x)/g'(x)$ does not exist. However,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1-x\sin(1/x)}{(\sin x)/x}=\frac{1}{1}=1.$$

4.10 Indeterminate forms $(0)(\infty)$

A product of functions fg *is said to have the indeterminate form* $0 \cdot \infty$ *as* $x \to b-$ if one of the factors approaches 0 and the other approaches $\pm\infty$ as $x \to b-$.

In this case, it may be useful to apply L'Hospital's rule after writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)},$$

since one of these ratios is of the form $0/0$ and the other is of the form ∞/∞ as $x \to b-$. Similar statements apply to limits as $x \to b+$, $x \to b$, and $x \to \pm\infty$.

Example: Evaluate the limit

$$\lim_{x\to 0+} x \log x.$$

Solution: The product $x \log x$ is of the form $0 \cdot \infty$ as $x \to 0+$. Converting it to an ∞/∞ form yields

$$\begin{aligned}
\lim_{x\to 0+} x \log x &= \lim_{x\to 0+} \frac{\log x}{1/x} \\
&= \lim_{x\to 0+} \frac{1/x}{-1/x^2} \\
&= -\lim_{x\to 0+} x = 0.
\end{aligned}$$

Example: Evaluate the limit

$$\lim_{x\to\infty} x \log 1 + 1/x.$$

Solution: Converting it to a $0/0$ form yields

$$\begin{aligned}
\lim_{x\to\infty} x \log(1+1/x) &= \lim_{x\to\infty} \frac{\log(1+1/x)}{1/x} \\
&= \lim_{x\to\infty} \frac{[1/(1+1/x)]\,(-1/x^2)}{-1/x^2} \\
&= \lim_{x\to\infty} \frac{1}{1+1/x} = 1.
\end{aligned}$$

In this case, converting to an ∞/∞ form complicates the problem:

$$\begin{aligned}
\lim_{x\to\infty} x \log(1+1/x) &= \lim_{x\to\infty} \frac{x}{1/\log(1+1/x)} \\
&= \lim_{x\to\infty} \frac{1}{\left(\frac{-1}{[\log(1+1/x)]^2}\right)\left(\frac{-1/x^2}{1+1/x}\right)} \\
&= \lim_{x\to\infty} x(x+1)[\log(1+1/x)]^2 = ?
\end{aligned}$$

4.10.1 Indeterminate form $\infty - \infty$

A difference $f - g$ *is of the form* $\infty - \infty$ *as* $x \to b-$ if

$$\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = \pm\infty.$$

In this case, it may be possible to manipulate $f - g$ into an expression that is no longer indeterminate, or is of the form $0/0$ or ∞/∞ as $x \to b-$.

Similar remarks apply to limits as $x \to b+$, $x \to b$, or $x \to \pm\infty$.

Example: Evaluate the limit

$$\lim_{x \to 0} \frac{\sin x}{x^2} - \frac{1}{x}.$$

Solution: See Lecture.

Example: Evaluate the limit

$$\lim_{x \to \infty} x^2 - x.$$

Solution: See Lecture.

4.10.2 Indeterminate forms $0^0, 1^\infty, \infty^0$

The function f^g is defined by

$$f(x)^{g(x)} = e^{g(x)\log f(x)} = \exp(g(x)\log f(x))$$

for all x such that $f(x) > 0$.

Therefore, if f and g are defined and $f(x) > 0$ on an interval (a, b), implies that

$$\lim_{x \to b-} [f(x)]^{g(x)} = \exp\left(\lim_{x \to b-} g(x)\log f(x)\right) \tag{4.6}$$

if $\lim_{x \to b-} g(x)\log f(x)$ exists in the extended real number system.

If this limit is $\pm\infty$ then (4.6) is valid if we define $e^{-\infty} = 0$ and $e^{\infty} = \infty$.

Indeterminate forms $0^0, 1^\infty, \infty^0$: The product $g \log f$ can be of the form $0 \cdot \infty$ in three ways as $x \to b-$:

1. If $\lim_{x \to b-} g(x) = 0$ and $\lim_{x \to b-} f(x) = 0$.

2. If $\lim_{x \to b-} g(x) = \pm\infty$ and $\lim_{x \to b-} f(x) = 1$.

3. If $\lim_{x \to b-} g(x) = 0$ and $\lim_{x \to b-} f(x) = \infty$.

In these three cases, we say that f^g *is of the form* 0^0, 1^∞, *and* ∞^0, *respectively, as* $x \to b-$.

Similar definitions apply to limits as $x \to b+$, $x \to b$, and $x \to \pm\infty$.

Example: Evaluate the following limit

$$\lim_{x \to 0+} x^x.$$

Solution: See Lecture.

Example: Evaluate the following limit

$$\lim_{x \to 1} x^{1/(x-1)}.$$

Solution:

Since

$$x^{1/(x-1)} = \exp\left(\frac{\log x}{x-1}\right)$$

and

$$\lim_{x \to 1} \frac{\log x}{x-1} = \lim_{x \to 1} \frac{1/x}{1} = 1,$$

it follows that

$$\lim_{x \to 1} x^{1/(x-1)} = e^1 = e.$$

Example: Evaluate the following limit

$$\lim_{x \to \infty} x^{1/x}.$$

Solution:

Since

$$x^{1/x} = \exp\left(\frac{\log x}{x}\right)$$

and

$$\lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0,$$

4.11 Taylor's Theorem

Recall the following lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

To generalize this result, we first restate it: the polynomial

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

which is of degree ≤ 1 and satisfies

$$T_1(x_0) = f(x_0), \quad T_1'(x_0) = f'(x_0).$$

$T_1(x)$ approximates f so well near x_0 that

$$\lim_{x \to x_0} \frac{f(x) - T_1(x)}{x - x_0} = 0.$$

Taylor polynomial: Suppose that f has n derivatives at x_0 and T_n is the polynomial of degree $\le n$ such that

$$T_n^{(r)}(x_0) = f^{(r)}(x_0), \quad 0 \le r \le n.$$

How well does T_n approximate f near x_0?
Since T_n is a polynomial of degree $\le n$, it can be written as

$$T_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n,$$

where $a_0, \ldots, a_n$ are constants.
Differentiating yields

$$T_n^{(r)}(x_0) = r! a_r, \quad 0 \le r \le n,$$

We obtained a_r uniquely as

$$a_r = \frac{f^{(r)}(x_0)}{r!}, \quad 0 \le r \le n.$$

Therefore,

$$\begin{aligned} T_n(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r. \end{aligned}$$

We call T_n *the nthTaylor polynomial of* f *about* x_0.

Theorem: If $f^{(n)}(x_0)$ exists for some integer $n \geq 1$ and T_n is the nth Taylor polynomial of f about x_0, then

$$\lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = 0.$$

Proof: The proof is by induction. Let P_n be the assertion of the theorem. If $n = 1$; that is, P_1 is true. Suppose that P_n is true for some integer $n \geq 1$, and $f^{(n+1)}$ exists.

Since the ratio

$$\frac{f(x) - T_{n+1}(x)}{(x - x_0)^{n+1}}$$

is indeterminate of the form $0/0$ as $x \to x_0$, L'Hospital's rule implies that

$$\lim_{x \to x_0} \frac{f(x) - T_{n+1}(x)}{(x - x_0)^{n+1}} = \frac{1}{n+1} \lim_{x \to x_0} \frac{f'(x) - T'_{n+1}(x)}{(x - x_0)^n} \tag{4.7}$$

if the limit on the right exists.

But f' has an nth derivative at x_0, and

$$T'_{n+1}(x) = \sum_{r=0}^{n} \frac{f^{(r+1)}(x_0)}{r!}(x - x_0)^r$$

is the nth Taylor polynomial of f' about x_0.

Therefore, the induction assumption, applied to f', implies that

$$\lim_{x \to x_0} \frac{f'(x) - T'_{n+1}(x)}{(x - x_0)^n} = 0.$$

This and (4.7) imply that

$$\lim_{x \to x_0} \frac{f(x) - T_{n+1}(x)}{(x - x_0)^{n+1}} = 0,$$

which completes the induction.

Lemma: If $f^{(n)}(x_0)$ exists, then

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r + E_n(x)(x - x_0)^n, \tag{4.8}$$

where

$$\lim_{x \to x_0} E_n(x) = E_n(x_0) = 0.$$

Proof: Define

$$E_n(x) = \begin{cases} \frac{f(x)-T_n(x)}{(x-x_0)^n}, & x \in D_f - \{x_0\}, \\ 0, & x = x_0. \end{cases}$$

Then we know that $\lim_{x \to x_0} E_n(x) = E_n(x_0) = 0$, and it is straightforward to verify (4.8).

Example: If $f(x) = e^x$, then $f^{(n)}(x) = e^x$.

Therefore, $f^{(n)}(0) = 1$ for $n \geq 0$, so the nth Taylor polynomial of f about $x_0 = 0$ is

$$T_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

According to approximation Theorem we have

$$\lim_{x \to 0} \frac{e^x - \sum_{r=0}^{n} \frac{x^r}{r!}}{x^n} = 0.$$

Example: If $f(x) = \log x$, then $f(1) = 0$ and

$$f^{(r)}(x) = (-1)^{(r-1)} \frac{(r-1)!}{x^r}, \quad r \geq 1.$$

so the nth Taylor polynomial of f about $x_0 = 1$ is

$$T_n(x) = \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r}(x-1)^r$$

if $n \geq 1$. ($T_0 = 0$.)

According to approximation Theorem we have

$$\lim_{x \to 1} \frac{\log x - \sum_{r=1}^{n} (-1)^{r-1} r (x-1)^r}{(x-1)^n} = 0, \quad n \geq 1.$$

Example: If $f(x) = (1+x)^q$, then

$$\begin{aligned} f'(x) &= q(1+x)^{q-1}, \quad f''(x) = q(q-1)(1+x)^{q-2} \\ &\vdots \\ f^{(n)}(x) &= q(q-1)\cdots(q-n+1)(1+x)^{q-n}. \end{aligned}$$

If we define

$$\binom{q}{0} = 1 \quad \text{and} \quad \binom{q}{n} = \frac{q(q-1)\cdots(q-n+1)}{n!}, \quad n \geq 1,$$

then

$$\frac{f^{(n)}(0)}{n!} = \binom{q}{n},$$

and the nth Taylor polynomial of f about 0 can be written as

$$T_n(x) = \sum_{r=0}^{n} \binom{q}{r} x^r.$$

According to approximation Theorem we have

$$\lim_{x \to 0} \frac{(1+x)^q - \sum_{r=0}^{n} \binom{q}{r} x^r}{x^n} = 0,$$

for $n \geq 0$.

Theorem: Suppose that f has n derivatives at x_0 and n is the smallest positive integer such that $f^{(n)}(x_0) \neq 0$.

1. If n is odd, x_0 is not a local extreme point of f.
2. If n is even, x_0 is a local maximum of f if $f^{(n)}(x_0) < 0$, or a local mininum of f if $f^{(n)}(x_0) > 0$.

Proof: Since $f^{(r)}(x_0) = 0$ for $1 \leq r \leq n-1$, we have

$$f(x) - f(x_0) = \left[\frac{f^{(n)}(x_0)}{n!} + E_n(x) \right] (x - x_0)^n,$$

in some interval containing x_0.

Since $\lim_{x \to x_0} E_n(x) = 0$ and $f^{(n)}(x_0) \neq 0$, there is a $\delta > 0$ such that

$$|E_n(x)| < \left| \frac{f^{(n)}(x_0)}{n!} \right| \quad \text{if} \quad |x - x_0| < \delta.$$

We can conclude that

$$\frac{f(x) - f(x_0)}{(x - x_0)^n}$$

has the same sign as $f^{(n)}(x_0)$ if $0 < |x - x_0| < \delta$.

If n is odd the denominator changes sign in every neighborhood of x_0, and therefore so must the numerator (since the ratio has constant sign for $0 < |x - x_0| < \delta$).

Consequently, $f(x_0)$ cannot be a local extreme value of f. This proves (1).

If n is even, the denominator is positive for $x \neq x_0$, so $f(x) - f(x_0)$ must have the same sign as $f^{(n)}(x_0)$ for $0 < |x - x_0| < \delta$. This proves part (2).

Example: Investigate the local extrema for the function $f(x) = e^{x^3}$.

Solution: For the given function we have $f'(x) = 3x^2e^{x^3}$, and 0 is the only critical point of f.

Since

$$\begin{aligned} f''(x) &= (6x + 9x^4)e^{x^3} \\ \text{and} \\ f'''(x) &= (6 + 54x^3 + 27x^6)e^{x^3}, \end{aligned}$$

$f''(0) = 0$ and $f'''(0) \neq 0$.

Therefore, 0 is not a local extreme point of f. Since f is differentiable everywhere, it has no local maxima or minima.

Example: Investigate the local extrema for the function $f(x) = \sin x^2$.

Solution: For the given function we have $f'(x) = 2x \cos x^2$, so the critical points of f are 0 and $\pm\sqrt{(k + 1/2)\pi}$, $k = 0, 1, 2, \ldots$.

Since

$$f''(x) = 2\cos x^2 - 4x^2 \sin x^2,$$

$$f''(0) = 2 \quad \text{and} \quad f''\left(\pm\sqrt{(k + 1/2)\pi)}\right) = (-1)^{k+1}(4k + 2)\pi.$$

Therefore, f attains local minima at 0 and $\pm\sqrt{(k + 1/2)\pi}$ for odd integers k, and local maxima at $\pm\sqrt{(k + 1/2)\pi}$ for even integers k.

Taylor's Theorem: Suppose that $f^{(n+1)}$ exists on an open interval I about x_0, and let x be in I.

Then the remainder

$$R_n(x) = f(x) - T_n(x)$$

can be written as

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1},$$

where c depends upon x and is between x and x_0.

Example: If $f(x) = e^x$, then $f'''(x) = e^x$, and Taylor's theorem with $n = 2$ implies that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{e^c x^3}{3!},$$

where c is between 0 and x. We have

$$e^{0.1} \approx T_2(0.1) = 1 + \frac{0.1}{1!} + \frac{(0.1)^2}{2!} = 1.105$$

hence

$$e^{0.1} = 1.105 + \frac{e^c(0.1)^3}{6},$$

where $0 < c < 0.1$.

Since $0 < e^c < e^{0.1}$, we know from this that

$$1.105 < e^{0.1} < 1.105 + \frac{e^{0.1}(0.1)^3}{6}.$$

Example: Since $0 < e^c < e^{0.1}$, we know from this that

$$1.105 < e^{0.1} < 1.105 + \frac{e^{0.1}(0.1)^3}{6}.$$

The second inequality implies that

$$e^{0.1}\left[1 - \frac{(0.1)^3}{6}\right] < 1.105,$$

so

$$e^{0.1} < 1.1052.$$

Therefore,

$$1.105 < e^{0.1} < 1.1052,$$

and the error in approximation is less than 0.0002.

CHAPTER 5

Riemann Integration

5.1 Riemann Sums

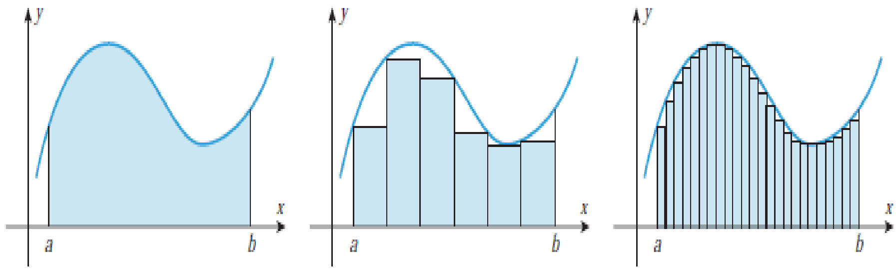


Figure 5.1: Area under the curve

Partition of an interval: Let f be a real valued function defined on a finite interval $[a, b]$.

A *partition of* $[a, b]$ is a set of subintervals

$$[x_0, x_1],\ \ [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

where

$$a = x_0 < x_1 \cdots < x_n = b.$$

Thus, any set of $n+1$ points satisfying $a = x_0 < x_1 \cdots < x_n = b$. defines a partition P of $[a, b]$, which we denote by

$$P = \{x_0, x_1, \ldots, x_n\}.$$

The points x_0, x_1, $\ldots$, x_n are the *partition points* of P.

The largest of the lengths of the subintervals $[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n]$, is the *norm* of P, written as $\|P\|$; thus,

$$\|P\| = \max_{1 \leq i \leq n} (x_i - x_{i-1}).$$

Refinement of a partition: If P and P' are partitions of $[a, b]$, then P' is a *refinement of* P if every partition point of P is also a partition point of P'; that is, if P' is obtained by inserting additional points between those of P.

Riemann sums: If f is defined on $[a, b]$, then a sum

$$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}),$$

where

$$x_{j-1} \leq c_j \leq x_j, \quad 1 \leq j \leq n,$$

is a *Riemann sum of f over the partition* $P = \{x_0, x_1, \dots, x_n\}$. (Occasionally we will say more simply that σ is a Riemann sum of f over $[a, b]$.)

Since c_j can be chosen arbitrarily in $[x_j, x_{j-1}]$, there are infinitely many Riemann sums for a given function f over a given partition P.

5.2 Riemann Integral

Let f be defined on $[a, b]$. We say that f is *Riemann integrable on* $[a, b]$ if there is a number L with the following property:

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|\sigma - L| < \varepsilon.$$

If σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$. In this case, we say that L is *the Riemann integral of f over* $[a, b]$, and write

$$\int_a^b f(x)\, dx = L.$$

Theorem: Prove that the Riemann integral $\int_a^b f(x)dx$, if it exists, is unique.

Proof: See Lecture.

- Prove: If $\int_a^b f(x)\, dx$ exists, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\sigma_1 - \sigma_2| < \varepsilon$ if σ_1 and σ_2 are Riemann sums of f over partitions P_1 and P_2 of $[a, b]$ with norms less than δ.

Example: Determine f is Riemann integrable over the given interval or not? Find $\int_a^b f(x)dx$, where

$$f(x) = 1, \quad a \leq x \leq b.$$

Solution: For the given function, we have

$$\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} (x_j - x_{j-1}).$$

Most of the terms in the sum on the right cancel in pairs; that is,

$$\begin{aligned} & \sum_{j=1}^{n}(x_j - x_{j-1}) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ = \quad & -x_0 + (x_1 - x_1) + (x_2 - x_2) + \cdots + (x_{n-1} - x_{n-1}) + x_n \\ = \quad & x_n - x_0 = b - a. \end{aligned}$$

Thus, every Riemann sum of f over any partition of $[a, b]$ equals $b - a$, so

$$\int_a^b dx = b - a.$$

Example: For the function

$$f(x) = x, \quad a \le x \le b,$$

Riemann sums are of the form

$$\sigma = \sum_{j=1}^{n} c_j (x_j - x_{j-1}). \tag{5.1}$$

Since $x_{j-1} \le c_j \le x_j$ and $(x_j + x_{j-1})/2$ is the midpoint of $[x_{j-1}, x_j]$, we can write

$$c_j = \frac{x_j + x_{j-1}}{2} + d_j, \tag{5.2}$$

where

$$|d_j| \le \frac{x_j - x_{j-1}}{2} \le \frac{\|P\|}{2}. \tag{5.3}$$

Example: Substituting (5.2) into (5.1) yields

$$\begin{aligned} \sigma &= \textstyle\sum_{j=1}^{n} \frac{x_j + x_{j-1}}{2}(x_j - x_{j-1}) + \sum_{j=1}^{n} d_j (x_j - x_{j-1}) \\ &= \textstyle\frac{1}{2}\sum_{j=1}^{n}(x_j^2 - x_{j-1}^2) + \sum_{j=1}^{n} d_j (x_j - x_{j-1}). \end{aligned} \tag{5.4}$$

Because of cancelations, we have

$$\sum_{j=1}^{n}(x_j^2 - x_{j-1}^2) = b^2 - a^2,$$

so (5.4) can be rewritten as

$$\sigma = \frac{b^2 - a^2}{2} + \sum_{j=1}^{n} d_j (x_j - x_{j-1}).$$

Hence,

$$\begin{aligned}\left|\sigma - \frac{b^2-a^2}{2}\right| &\leq \sum_{j=1}^{n} |d_j|(x_j - x_{j-1}) \leq \frac{\|P\|}{2}\sum_{j=1}^{n}(x_j - x_{j-1}) \\ &= \frac{\|P\|}{2}(b-a).\end{aligned}$$

Therefore, every Riemann sum of f over a partition P of $[a,b]$ satisfies

$$\left|\sigma - \frac{b^2-a^2}{2}\right| < \varepsilon \quad \text{if} \quad \|P\| < \delta = \frac{2\varepsilon}{b-a}.$$

Hence,

$$\int_a^b x\,dx = \frac{b^2-a^2}{2}.$$

Theorem: If f is unbounded on $[a,b]$, then f is not integrable on $[a,b]$.

Proof: We will show that if f is unbounded on $[a,b]$, P is any partition of $[a,b]$, and $M > 0$, then there are Riemann sums σ and σ' of f over P such that

$$|\sigma - \sigma'| \geq M. \tag{5.5}$$

Let

$$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$$

be a Riemann sum of f over a partition P of $[a,b]$.

There must be an integer i in $\{1, 2, \ldots, n\}$ such that

$$|f(c) - f(c_i)| \geq \frac{M}{x_i - x_{i-1}}$$

for some c in $[x_{i-1}x_i]$.

Because if there were not so, we would have

$$|f(x) - f(c_j)| < \frac{M}{x_j - x_{j-1}}, \quad x_{j-1} \leq x \leq x_j, \quad 1 \leq j \leq n.$$

Then

$$\begin{aligned}|f(x)| &= |f(c_j) + f(x) - f(c_j)| \leq |f(c_j)| + |f(x) - f(c_j)| \\ &\leq |f(c_j)| + \frac{M}{x_j - x_{j-1}}, \quad x_{j-1} \leq x \leq x_j, \quad 1 \leq j \leq n.\end{aligned}$$

which implies that

$$|f(x)| \leq \max_{1 \leq j \leq n} |f(c_j)| + \frac{M}{x_j - x_{j-1}}, \quad a \leq x \leq b,$$

contradicting the assumption that f is unbounded on $[a, b]$.

Consider the Riemann sum

$$\sigma' = \sum_{j=1}^{n} f(c_j')(x_j - x_{j-1})$$

over the same partition P, where

$$c_j' = \begin{cases} c_j, & j \neq i, \\ c, & j = i. \end{cases}$$

Since

$$|\sigma - \sigma'| = |f(c) - f(c_i)|(x_i - x_{i-1}),$$

hence due to

$$|f(c) - f(c_i)| \geq \frac{M}{x_i - x_{i-1}}$$

implies (5.5).

5.3 Upper and Lower Integrals

Upper and lower integrals: If f is bounded on $[a, b]$ and $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, let

$$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x) \quad \text{and} \quad m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x).$$

The *upper sum of f over P* is

$$S(P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}),$$

and the *upper integral of f over*, $[a, b]$, denoted by

$$\overline{\int_a^b} f(x)\, dx,$$

is the infimum of all upper sums.

Lower integrals: The *lower sum of f over P* is

$$s(P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),$$

and the *lower integral of f over* $[a, b]$, denoted by

$$\underline{\int_a^b} f(x)\, dx,$$

is the supremum of all lower sums.

Remark: If $m \le f(x) \le M$ for all x in $[a,b]$, then

$$m(b-a) \le s(P) \le S(P) \le M(b-a)$$

for every partition P. Thus, the set of upper sums of f over all partitions P of $[a,b]$ is bounded, as is the set of lower sums.

Therefore, definition of infimum and supremum imply that $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$ exist, are unique, and satisfy the inequalities

$$m(b-a) \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$

and

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le M(b-a).$$

Theorem: Let f be bounded on $[a,b]$, and let P be a partition of $[a,b]$. Then

1. The upper sum $S(P)$ of f over P is the supremum of the set of all Riemann sums of f over P.

2. The lower sum $s(P)$ of f over P is the infimum of the set of all Riemann sums of f over P.

Proof: If $P = \{x_0, x_1, \ldots, x_n\}$, then

$$S(P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}),$$

where

$$M_j = \sup_{x_{j-1} \le x \le x_j} f(x).$$

An arbitrary Riemann sum of f over P is of the form

$$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}),$$

where $x_{j-1} \le c_j \le x_j$.

Since $f(c_j) \le M_j$, it follows that $\sigma \le S(P)$. Now let $\varepsilon > 0$ and choose $\overline{c}_j$ in $[x_{j-1}, x_j]$ so that

$$f(\overline{c}_j) > M_j - \frac{\varepsilon}{n(x_j - x_{j-1})}, \quad 1 \le j \le n.$$

The Riemann sum produced in this way is

$$\begin{aligned}\overline{\sigma} &= \sum_{j=1}^{n} f(\overline{c}_j)(x_j - x_{j-1}) > \sum_{j=1}^{n}\left[M_j - \frac{\varepsilon}{n(x_j - x_{j-1})})\right](x_j - x_{j-1}) \\ &= S(P) - \varepsilon.\end{aligned}$$

Now from definition $S(P)$ is the supremum of the set of Riemann sums of f over P.

Example: Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since every interval contains both rational and irrational numbers

$$m_j = 0 \quad \text{and} \quad M_j = 1, \quad 1 \le j \le n.$$

Hence,

$$\begin{aligned} S(P) &= \sum_{j=1}^{n} 1 \cdot (x_j - x_{j-1}) = b - a \\ s(P) &= \sum_{j=1}^{n} 0 \cdot (x_j - x_{j-1}) = 0. \end{aligned}$$

Since all upper sums equal $b - a$ and all lower sums equal 0, from definition we have

$$\overline{\int_a^b} f(x)\,dx = b - a \quad \text{and} \quad \underline{\int_a^b} f(x)\,dx = 0.$$

Example: Let f be defined on $[1, 2]$ by $f(x) = 0$ if x is irrational and $f(p/q) = 1/q$ if p and q are positive integers with no common factors.

If $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[1, 2]$, then $m_j = 0$, $1 \le j \le n$, so $s(P) = 0$; hence,

$$\underline{\int_1^2} f(x)\,dx = 0.$$

We now show that

$$\overline{\int_1^2} f(x)\,dx = 0 \tag{5.6}$$

also.

Example: Since $S(P) > 0$ for every P, from definition we have

$$\overline{\int_1^2} f(x)\,dx \geq 0.$$

We need only show that

$$\overline{\int_1^2} f(x)\,dx \leq 0,$$

which will follow if we show that no positive number is less than every upper sum.

To this end, we observe that if $0 < \varepsilon < 2$, then $f(x) \geq \varepsilon/2$ for only finitely many values of x in $[1, 2]$.

Let k be the number of such points and let P_0 be a partition of $[1, 2]$ such that

$$\|P_0\| < \frac{\varepsilon}{2k}. \tag{5.7}$$

Example: Consider the upper sum

$$S(P_0) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}).$$

There are at most k values of j in this sum for which $M_j \geq \varepsilon/2$, and $M_j \leq 1$ even for these. The contribution of these terms to the sum is less than $k(\varepsilon/2k) = \varepsilon/2$, because of (5.7).

Since $M_j < \varepsilon/2$ for all other values of j, the sum of the other terms is less than

$$\frac{\varepsilon}{2}\sum_{j=1}^{n}(x_j - x_{j-1}) = \frac{\varepsilon}{2}(x_n - x_0) = \frac{\varepsilon}{2}(2-1) = \frac{\varepsilon}{2}.$$

Therefore, $S(P_0) < \varepsilon$ and, since ε can be chosen as small as we wish, no positive number is less than all upper sums.

Lemma: Suppose that

$$|f(x)| \leq M, \quad a \leq x \leq b,$$

and let P' be a partition of $[a, b]$ obtained by adding r points to a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$.

Then

$$\begin{aligned} S(P) \geq S(P') \quad &\geq \quad S(P) - 2Mr\|P\| \\ &\text{and} \\ s(P) \leq s(P') \quad &\leq \quad s(P) + 2Mr\|P\|. \end{aligned}$$

Proof: First suppose that $r = 1$, so P' is obtained by adding one point c to the partition $P = \{x_0, x_1, \dots, x_n\}$; then $x_{i-1} < c < x_i$ for some i in $\{1, 2, \dots, n\}$.

If $j \neq i$, the product $M_j(x_j - x_{j-1})$ appears in both $S(P)$ and $S(P')$ and cancels out of the difference $S(P) - S(P')$.

Therefore, if

$$M_{i1} = \sup_{x_{i-1} \leq x \leq c} f(x) \quad \text{and} \quad M_{i2} = \sup_{c \leq x \leq x_i} f(x),$$

then

$$\begin{aligned} S(P) - S(P') &= M_i(x_i - x_{i-1}) - M_{i1}(c - x_{i-1}) - M_{i2}(x_i - c) \\ &= (M_i - M_{i1})(c - x_{i-1}) + (M_i - M_{i2})(x_i - c). \end{aligned} \tag{5.8}$$

Since f is bounded implies that

$$0 \leq M_i - M_{ir} \leq 2M, \quad r = 1, 2,$$

(5.8) implies that

$$0 \leq S(P) - S(P') \leq 2M(x_i - x_{i-1}) \leq 2M\|P\|.$$

This proves (5.11) for $r = 1$.

Now suppose that $r > 1$ and P' is obtained by adding points $c_1, c_2, \dots, c_r$ to P.

Let $P^{(0)} = P$ and, for $j \geq 1$, let $P^{(j)}$ be the partition of $[a, b]$ obtained by adding c_j to $P^{(j-1)}$. Then the result just proved implies that

$$0 \leq S(P^{(j-1)}) - S(P^{(j)}) \leq 2M\|P^{(j-1)}\|, \quad 1 \leq j \leq r.$$

Adding these inequalities and taking account of cancellations yields

$$0 \leq S(P^{(0)}) - S(P^{(r)}) \leq 2M(\|P^{(0)}\| + \|P^{(1)}\| + \cdots + \|P^{(r-1)}\|). \tag{5.9}$$

Since $P^{(0)} = P$, $P^{(r)} = P'$, and $\|P^{(k)}\| \leq \|P^{(k-1)}\|$ for $1 \leq k \leq r - 1$, (5.9) implies that

$$0 \leq S(P) - S(P') \leq 2Mr\|P\|,$$

which is equivalent to (5.11).

Theorem: If f is bounded on $[a, b]$, then

$$\underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx. \tag{5.10}$$

Proof: Suppose that P_1 and P_2 are partitions of $[a, b]$ and P' is a refinement of both.

We have proved

$$\begin{array}{rcl} S(P) \geq S(P') & \geq & S(P) - 2Mr\|P\| \quad (*) \\ \text{and} & & \\ s(P) \leq s(P') & \leq & s(P) + 2Mr\|P\| \quad (**). \end{array}$$

Letting $P = P_1$ in (**) and $P = P_2$ in (*) shows that

$$s(P_1) \leq s(P') \quad \text{and} \quad S(P') \leq S(P_2).$$

Since $s(P') \leq S(P')$, this implies that $s(P_1) \leq S(P_2)$. Thus, every lower sum is a lower bound for the set of all upper sums.

Since $\overline{\int_a^b} f(x)\,dx$ is the infimum of this set, it follows that

$$s(P_1) \leq \overline{\int_a^b} f(x)\,dx$$

for every partition P_1 of $[a,b]$.

This means that $\overline{\int_a^b} f(x)\,dx$ is an upper bound for the set of all lower sums.

Since $\underline{\int_a^b} f(x)\,dx$ is the supremum of this set, this implies (5.10).

Theorem: If f is integrable on $[a,b]$, then

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Proof: Suppose that P is a partition of $[a,b]$ and σ is a Riemann sum of f over P. Since

$$\begin{aligned} \overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx &= \left(\overline{\int_a^b} f(x)\,dx - S(P)\right) + (S(P) - \sigma) \\ &\quad + \left(\sigma - \int_a^b f(x)\;dx\right). \end{aligned}$$

The triangle inequality implies that

$$\begin{aligned} \left|\overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| &\leq \left|\overline{\int_a^b} f(x)\,dx - S(P)\right| + |S(P) - \sigma| \\ &\quad + \left|\sigma - \int_a^b f(x)\;dx\right|. \end{aligned} \tag{5.11}$$

Now suppose that $\varepsilon > 0$. From definition of Riemann upper sum, there is a partition P_0 of $[a,b]$ such that

$$\overline{\int_a^b} f(x)\,dx \leq S(P_0) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{3}. \tag{5.12}$$

From definition of Riemann integral, there is a $\delta > 0$ such that

$$\left|\sigma - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{3}. \tag{5.13}$$

If $\|P\| < \delta$ and P is a refinement of P_0.
Since $S(P) \le S(P_0)$ by Lemma we have proved, (5.12) implies that

$$\overline{\int_a^b} f(x)\,dx \le S(P) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{3},$$

so

$$\left|S(P) - \overline{\int_a^b} f(x)\,dx\right| < \frac{\varepsilon}{3} \tag{5.14}$$

in addition to (5.13). Now (5.11), (5.13), and (5.14) imply that

$$\left|\overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{2\varepsilon}{3} + |S(P) - \sigma| \tag{5.15}$$

for every Riemann sum σ of f over P.
Since $S(P)$ is the supremum of these Riemann sums, we may choose σ so that

$$|S(P) - \sigma| < \frac{\varepsilon}{3}.$$

Now (5.15) implies that

$$\left|\overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \varepsilon.$$

Since ε is an arbitrary positive number, it follows that

$$\overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Lemma: If f is bounded on $[a, b]$ and $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\overline{\int_a^b} f(x)\,dx \le S(P) < \overline{\int_a^b} f(x)\,dx + \varepsilon \tag{5.16}$$

and

$$\underline{\int_a^b} f(x)\,dx \ge s(P) > \underline{\int_a^b} f(x)\,dx - \varepsilon$$

if $\|P\| < \delta$.

Proof: The first inequality follows immediately from definition of Riemann upper integral.

To establish the second inequality, suppose that $|f(x)| \le K$ if $a \le x \le b$. From definition of the Riemann upper integral, there is a partition $P_0 = \{x_0, x_1, \ldots, x_{r+1}\}$ of $[a, b]$ such that

$$S(P_0) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}. \tag{5.17}$$

If P is any partition of $[a, b]$, let P' be constructed from the partition points of P_0 and P. Then we know that

$$S(P') \le S(P_0). \tag{5.18}$$

Since P' is obtained by adding at most r points to P, we have

$$S(P') \ge S(P) - 2Kr\|P\|. \tag{5.19}$$

Now (5.17), (5.18), and (5.19) imply that

$$\begin{aligned} S(P) &\le S(P') + 2Kr\|P\| \\ &\le S(P_0) + 2Kr\|P\| \\ &< \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} + 2Kr\|P\|. \end{aligned}$$

Therefore, (5.16) holds if

$$\|P\| < \delta = \frac{\varepsilon}{4Kr}.$$

Theorem: If f is bounded on $[a, b]$ and

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = L,$$

then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = L.$$

Proof: If $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\underline{\int_a^b} f(x)\,dx - \varepsilon < s(P) \le S(P) < \overline{\int_a^b} f(x)\,dx + \varepsilon, \tag{5.20}$$

If $\|P\| < \delta$.

If σ is a Riemann sum of f over P, then

$$s(P) \le \sigma \le S(P).$$

So (5.21) and (5.20) imply that

$$L - \varepsilon < \sigma < L + \varepsilon$$

if $\|P\| < \delta$.

Recall the following theorems:
Theorem: If f is integrable on $[a, b]$, then

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Theorem: If f is bounded on $[a, b]$ and

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = L, \tag{5.21}$$

then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = L. \tag{5.22}$$

Theorem: A bounded function f is integrable on $[a, b]$ if and only if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Theorem: If f is bounded on $[a, b]$, then f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$ there is a partition P of $[a, b]$ for which

$$S(P) - s(P) < \varepsilon.$$

Proof: Since

$$s(P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le S(P)$$

for all P.

The inequality $S(P) - s(P) < \varepsilon$, implies that

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon.$$

Since ε can be any positive number, this implies that

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Therefore, $\int_a^b f(x)\,dx$ exists.

Since ε can be any positive number, this implies that

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

For the converse of the theorem try yourself

Theorem: If f is bounded on $[a,b]$, then f is integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there is a partition P of $[a,b]$ for which

$$S(P) - s(P) < \varepsilon.$$

Theorem: If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$.

Since f is continuous on $[a,b]$, there are points c_j and c_j' in $[x_{j-1}, x_j]$ such that

$$f(c_j) = M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$

and

$$f(c_j') = m_j = \inf_{x_{j-1} \le x \le x_j} f(x).$$

Therefore,

$$S(P) - s(P) = \sum_{j=1}^{n} \left[f(c_j) - f(c_j') \right] (x_j - x_{j-1}). \tag{5.23}$$

Since f is uniformly continuous on $[a,b]$, there is for each $\varepsilon > 0$ a $\delta > 0$ such that

$$|f(x') - f(x)| < \frac{\varepsilon}{b-a}.$$

If x and x' are in $[a,b]$ and $|x - x'| < \delta$. If $\|P\| < \delta$, then $|c_j - c_j'| < \delta$ and, from (5.23),

$$S(P) - s(P) < \frac{\varepsilon}{b-a} \sum_{j=1}^{n} (x_j - x_{j-1}) = \varepsilon.$$

Hence, f is integrable on $[a,b]$.

Theorem: If f is bounded on $[a, b]$, then f is integrable on $[a, b]$ if and only if for each $\varepsilon > 0$ there is a partition P of $[a, b]$ for which

$$S(P) - s(P) < \varepsilon.$$

Theorem: If f is monotonic on $[a, b]$, then f is integrable on $[a, b]$.

Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since f is nondecreasing,

$$\begin{aligned} f(x_j) &= M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x) \\ f(x_{j-1}) &= m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x). \end{aligned}$$

Hence,

$$S(P) - s(P) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))(x_j - x_{j-1}).$$

Since $0 < x_j - x_{j-1} \leq \|P\|$ and $f(x_j) - f(x_{j-1}) \geq 0$,

$$\begin{aligned} S(P) - s(P) &\leq \|P\| \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) \\ &= \|P\|(f(b) - f(a)). \end{aligned}$$

Therefore,

$$S(P) - s(P) < \varepsilon \quad \text{if} \quad \|P\|(f(b) - f(a)) < \varepsilon,$$

so f is integrable on $[a, b]$.

Theorem: If f and g are integrable on $[a, b]$, then so is $f + g$, and

$$\int_a^b (f + g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Proof: Any Riemann sum of $f + g$ over a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ can be written as

$$\begin{aligned} \sigma_{f+g} &= \textstyle\sum_{j=1}^{n} [f(c_j) + g(c_j)](x_j - x_{j-1}) \\ &= \textstyle\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) + \sum_{j=1}^{n} g(c_j)(x_j - x_{j-1}) \\ &= \sigma_f + \sigma_g, \end{aligned}$$

where σ_f and σ_g are Riemann sums for f and g.

Definition of Riemann integral implies that if $\varepsilon > 0$ there are positive numbers δ_1 and δ_2 such that

$$\left|\sigma_f - \int_a^b f(x)\,dx\right| \quad < \quad \frac{\varepsilon}{2} \quad \text{if} \quad \|P\| < \delta_1$$

and

$$\left|\sigma_g - \int_a^b g(x)\,dx\right| \quad < \quad \frac{\varepsilon}{2} \quad \text{if} \quad \|P\| < \delta_2.$$

If $\|P\| < \delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} \left|\sigma_{f+g} - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| &= \left|\left(\sigma_f - \int_a^b f(x)\,dx\right) + \left(\sigma_g - \int_a^b g(x)\,dx\right)\right| \\ &\le \left|\sigma_f - \int_a^b f(x)\,dx\right| + \left|\sigma_g - \int_a^b g(x)\,dx\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

so the conclusion follows from Definition.

Example: Determine whether the function $f(x) = 1 + x$ is integrable on $[a, b]$?

Solution: If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x)) = \sin\frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$

Theorem: If f is integrable on $[a, b]$ and c is a constant, then cf is integrable on $[a, b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Proof: See Lecture.

Theorem: If f_1, f_2, $\dots$, f_n are integrable on $[a, b]$ and c_1, c_2, $\dots$, c_n are constants, then $c_1f_1 + c_2f_2 + \cdots + c_nf_n$ is integrable on $[a, b]$ and

$$\begin{aligned} \int_a^b (c_1f_1 + c_2f_2 + \cdots + c_nf_n)(x)\,dx &= c_1\int_a^b f_1(x)\,dx \\ &\quad + \cdots + c_n\int_a^b f_n(x)\,dx. \end{aligned}$$

Proof: See Lecture.

Theorem: If f and g are integrable on $[a,b]$ and $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof: Since $g(x) - f(x) \ge 0$, every lower sum of $g - f$ over any partition of $[a,b]$ is nonnegative.

Therefore,

$$\underline{\int_a^b} (g(x) - f(x))\,dx \ge 0.$$

Hence,

$$\begin{aligned} \int_a^b g(x)\,dx - \int_a^b f(x)\,dx &= \int_a^b (g(x) - f(x))\,dx \\ &= \underline{\int_a^b} (g(x) - f(x))\,dx \ge 0. \end{aligned} \tag{5.24}$$

which yields . (The first equality in (5.24) follows from Theorems **??** and **??**; the second, from Theorem **??**.)

Theorem: If f is integrable on $[a,b]$, then so is $|f|$, and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof: Let P be a partition of $[a,b]$ and define

$$\begin{aligned} M_j &= \sup\{f(x) : x_{j-1} \le x \le x_j\}, \\ m_j &= \inf\{f(x) : x_{j-1} \le x \le x_j\}, \\ \overline{M}_j &= \sup\{|f(x)| : x_{j-1} \le x \le x_j\}, \\ \overline{m}_j &= \inf\{|f(x)| : x_{j-1} \le x \le x_j\}. \end{aligned}$$

Then

$$\begin{aligned} \overline{M}_j - \overline{m}_j &= \sup\{|f(x)| - |f(x')| : x_{j-1} \le x, x' \le x_j\} \\ &\le \sup\{|f(x) - f(x')| : x_{j-1} \le x, x' \le x_j\} \\ &= M_j - m_j. \end{aligned} \tag{5.25}$$

Therefore,

$$\overline{S}(P) - \overline{s}(P) \le S(P) - s(P),$$

where the upper and lower sums on the left are associated with $|f|$ and those on the right are associated with f.

Suppose that $\varepsilon > 0$. Since f is integrable on $[a, b]$, there is a partition P of $[a, b]$ such that $S(P) - s(P) < \varepsilon$.

This inequality and (5.25) imply that $\overline{S}(P) - \overline{s}(P) < \varepsilon$. Therefore, $|f|$ is integrable on $[a, b]$.

Since

$$f(x) \leq |f(x)| \quad \text{and} \quad -f(x) \leq |f(x)|, \quad a \leq x \leq b,$$

We have

$$\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \quad \text{and} \quad -\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx,$$

hence the result.

Theorem: If f and g are integrable on $[a, b]$, then so is the product fg.

Proof: We consider the case where f and g are nonnegative, and leave the rest of the proof to you. The subscripts f, g, and fg in the following argument identify the functions with which the various quantities are associated. We assume that neither f nor g is identically zero on $[a, b]$, since the conclusion is obvious if one of them is.

If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then

$$S_{fg}(P) - s_{fg}(p) = \sum_{j=1}^{n} (M_{fg,j} - m_{fg,j})(x_j - x_{j-1}). \tag{5.26}$$

Since f and g are nonnegative, $M_{fg,j} \leq M_{f,j}M_{g,j}$ and $m_{fg,j} \geq m_{f,j}m_{g,j}$. Hence,

$$\begin{aligned} M_{fg,j} - m_{fg,j} &\leq M_{f,j}M_{g,j} - m_{f,j}m_{g,j} \\ &= (M_{f,j} - m_{f,j})M_{g,j} + m_{f,j}(M_{g,j} - m_{g,j}) \\ &\leq M_g(M_{f,j} - m_{f,j}) + M_f(M_{g,j} - m_{g,j}), \end{aligned}$$

where M_f and M_g are upper bounds for f and g on $[a, b]$.

From (5.26) and the last inequality,

$$S_{fg}(P) - s_{fg}(P) \leq M_g[S_f(P) - s_f(P)] + M_f[S_g(P) - s_g(P)]. \tag{5.27}$$

Now suppose that $\varepsilon > 0$. There are partitions P_1 and P_2 of $[a, b]$ such that

$$S_f(P_1) - s_f(P_1) < \frac{\varepsilon}{2M_g} \quad \text{and} \quad S_g(P_2) - s_g(P_2) < \frac{\varepsilon}{2M_f}. \tag{5.28}$$

If P is a refinement of both P_1 and P_2, then (5.28) and we can write

$$S_f(P) - s_f(P) < \frac{\varepsilon}{2M_g} \quad \text{and} \quad S_g(P) - s_g(P) < \frac{\varepsilon}{2M_f}.$$

This and (5.27) yield

$$S_{fg}(P) - s_{fg}(P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, fg is integrable on $[a, b]$, by Theorem **??**.

Theorem: Suppose that u is continuous and v is integrable and nonnegative on $[a, b]$. Then

$$\int_a^b u(x)v(x)\,dx = u(c)\int_a^b v(x)\,dx \tag{5.29}$$

for some c in $[a, b]$.

Proof: Since u is continuous implies u is integrable on $[a, b]$. The function uv is integrable on $[a, b]$.

If $m = \min\{u(x) : a \le x \le b\}$ and $M = \max\{u(x) : a \le x \le b\}$, then

$$m \le u(x) \le M$$

and, since $v(x) \ge 0$,

$$mv(x) \le u(x)v(x) \le Mv(x).$$

Therefore, we can write

$$m\int_a^b v(x)\,dx \le \int_a^b u(x)v(x)\,dx \le M\int_a^b v(x)\,dx. \tag{5.30}$$

This implies that (5.32) holds for any c in $[a, b]$ if $\int_a^b v(x)\,dx = 0$.

If $\int_a^b v(x)\,dx \ne 0$, let

$$\overline{u} = \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx} \tag{5.31}$$

Since $\int_a^b v(x)\,dx > 0$ in this case (why?), (5.33) implies that $m \le \overline{u} \le M$. The intermediate value theorem implies that $\overline{u} = u(c)$ for some c in $[a, b]$. This implies (5.32).

Theorem: Suppose that u is continuous and v is integrable and nonnegative on $[a, b]$. Then

$$\int_a^b u(x)v(x)\,dx = u(c)\int_a^b v(x)\,dx \tag{5.32}$$

for some c in $[a, b]$.

Proof: Since u is continuous implies u is integrable on $[a, b]$. The function uv is integrable on $[a, b]$.

If $m = \min\{u(x) : a \le x \le b\}$ and $M = \max\{u(x) : a \le x \le b\}$, then

$$m \le u(x) \le M$$

and, since $v(x) \geq 0$,

$$mv(x) \leq u(x)v(x) \leq Mv(x).$$

Proof: Therefore, we can write

$$m \int_a^b v(x)\, dx \leq \int_a^b u(x)v(x)\, dx \leq M \int_a^b v(x)\, dx. \tag{5.33}$$

This implies that (5.32) holds for any c in $[a, b]$ if $\int_a^b v(x)\, dx = 0$.

If $\int_a^b v(x)\, dx \neq 0$, let

$$\overline{u} = \frac{\int_a^b u(x)v(x)\, dx}{\int_a^b v(x)\, dx} \tag{5.34}$$

Since $\int_a^b v(x)\, dx > 0$ in this case (why?), (5.33) implies that $m \leq \overline{u} \leq M$. The intermediate value theorem implies that $\overline{u} = u(c)$ for some c in $[a, b]$. This implies (5.32).

Average of $u(x)$ on $[a, b]$ and weighted average : Recall the following

$$\overline{u} = \frac{\int_a^b u(x)v(x)\, dx}{\int_a^b v(x) dx}$$

If $v(x) = 1$, we have

$$\overline{u} = \frac{1}{b-a} \int_a^b u(x) dx,$$

is known as average of $u(x)$ over $[a, b]$.

Theorem: If f is integrable on $[a, b]$ and $a \leq a_1 < b_1 \leq b$, then f is integrable on $[a_1, b_1]$.

Proof: Suppose that $\varepsilon > 0$, there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that

$$S(P) - s(P) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \varepsilon. \tag{5.35}$$

We may assume that a_1 and b_1 are partition points of P, because if not they can be inserted to obtain a refinement P' such that $S(P') - s(P') \leq S(P) - s(P)$.

Let $a_1 = x_r$ and $b_1 = x_s$. Since every term in (5.35) is nonnegative,

$$\sum_{j=r+1}^{s} (M_j - m_j)(x_j - x_{j-1}) < \varepsilon.$$

Theorem: If f is integrable on $[a, b]$ and $a \leq a_1 < b_1 \leq b$, then f is integrable on $[a_1, b_1]$.

Proof: Thus, $\overline{P} = \{x_r, x_{r+1}, \ldots, x_s\}$ is a partition of $[a_1, b_1]$ over which the upper and lower sums of f satisfy

$$S(\overline{P}) - s(\overline{P}) < \varepsilon.$$

Therefore, f is integrable on $[a_1, b_1]$.

If f is integrable on $[a, b]$ and $a \leq a_1 < b_1 \leq b$, then f is integrable on $[a_1, b_1]$.

Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.36}$$

So far we have defined $\int_\alpha^\beta f(x)\,dx$ only for the case where $\alpha < \beta$.

If $\alpha < \beta$, we define

$$\int_\beta^\alpha f(x)\,dx = -\int_\alpha^\beta f(x)\,dx.$$

$$\int_\alpha^\alpha f(x)\,dx = 0.$$

Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.37}$$

Proof: See Lecture.

5.4 Fundamental Theorem of Calculus

Theorem: If f is integrable on $[a, b]$ and $a \leq c \leq b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a, b]$, and is therefore continuous on $[a, b]$.

Proof: If x and x' are in $[a, b]$, then

$$F(x) - F(x') = \int_c^x f(t)\,dt - \int_c^{x'} f(t)\,dt = \int_{x'}^x f(t)\,dt.$$

Since $|f(t)| \leq K$ $(a \leq t \leq b)$ for some constant K,

$$\left|\int_{x'}^x f(t)\,dt\right| \leq K|x - x'|, \quad a \leq x,\, x' \leq b.$$

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a,b]$, and is therefore continuous on $[a,b]$.

Proof: Consequently,

$$|F(x) - F(x')| \le K|x - x'|, \quad a \le x,\, x' \le b.$$

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

Proof: Since

$$\frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt = f(x_0).$$

We can write

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]\,dt.$$

From this

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \frac{1}{|x - x_0|}\left|\int_{x_0}^x |f(t) - f(x_0)|\,dt\right|.$$

Since f is continuous at x_0, there is for each $\varepsilon > 0$ a $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta$$

and t is between x and x_0.

Therefore, we have

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon\frac{|x - x_0|}{|x - x_0|} = \varepsilon \quad \text{if} \quad 0 < |x - x_0| < \delta.$$

Hence, $F'(x_0) = f(x_0)$.

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

Example: If

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ x+1, & 1 < x \le 2. \end{cases}$$

Then the function

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \frac{x^2}{2}, & 0 < x \le 1, \\ \frac{x^2}{2} + x - 1, & 1 < x \le 2, \end{cases}$$

is continuous on $[0,2]$.

We can conclude

$$F'(x) = \begin{cases} x = f(x), & 0 < x < 1, \\ x+1 = f(x), & 1 < x < 2. \end{cases}$$

$$\begin{aligned} F'_+(0) &= \lim_{x\to 0+} \frac{F(x)-F(0)}{x} = \lim_{x\to 0+} \frac{(x^2/2) - 0}{x} = 0 = f(0), \\ F'_-(2) &= \lim_{x\to 2-} \frac{F(x)-F(2)}{x-2} = \lim_{x\to 2-} \frac{(x^2/2)+x-1-3}{x-2} \\ &= \lim_{x\to 2-} \frac{x+4}{2} = 3 = f(2). \end{aligned}$$

F does not have a derivative at $x=1$, where f is discontinuous, since

$$F'_-(1) = 1 \quad \text{and} \quad F'_+(1) = 2.$$

Theorem: Suppose that F is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and f is integrable on $[a,b]$.

Suppose also that

$$F'(x) = f(x), \quad a < x < b.$$

Then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.38}$$

Proof: If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, then

$$F(b) - F(a) = \sum_{j=1}^{n} (F(x_j) - F(x_{j-1})). \tag{5.39}$$

From mean value theorem, there is in each open interval (x_{j-1}, x_j) a point c_j such that

$$F(x_j) - F(x_{j-1}) = f(c_j)(x_j - x_{j-1}).$$

Hence, (5.39) can be written as

$$F(b) - F(a) = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \sigma,$$

where σ is a Riemann sum for f over P.

Since f is integrable on $[a, b]$, there is for each $\varepsilon > 0$ a $\delta > 0$ such that

$$\left| \sigma - \int_a^b f(x)\, dx \right| < \varepsilon \quad \text{if} \quad \|P\| < \delta.$$

Therefore,

$$\left| F(b) - F(a) - \int_a^b f(x)\, dx \right| < \varepsilon$$

for every $\varepsilon > 0$.

Lemma: If f' is integrable on $[a, b]$, then

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Proof: See Lecture.

5.4.1 Anti-derivative of a Function

Anti-derivative of a function: A function F is an *antiderivative* of f on $[a, b]$ if F is continuous on $[a, b]$ and differentiable on (a, b), with

$$F'(x) = f(x), \quad a < x < b.$$

If F is an antiderivative of f on $[a, b]$, then so is $F + c$ for any constant c.

Conversely, if F_1 and F_2 are antiderivatives of f on $[a, b]$, then $F_1 - F_2$ is constant on $[a, b]$.

Theorem: If f is continuous on $[a, b]$, then f has an antiderivative on $[a, b]$. Moreover, if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof: The function $F_0(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$.

Furthermore, $F_0'(x) = f(x)$ on (a, b) by previous theorem.

Therefore, F_0 is an antiderivative of f on $[a, b]$.

Now let $F = F_0 + c$ (c = constant) be an arbitrary antiderivative of f on $[a, b]$. Then

$$F(b) - F(a) = \int_a^b f(x)\,dx + c - \int_a^a f(x)\,dx - c = \int_a^b f(x)\,dx.$$

When applying this theorem, we will use the familiar notation

$$F(b) - F(a) = F(x)\Big|_a^b.$$

5.5 Integration by Parts

Theorem: If u' and v' are integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx. \tag{5.40}$$

Proof: Since u and v are continuous on $[a, b]$, they are integrable on $[a, b]$.

Therefore, using the product of two integrable function is integrable and sum of two integrable functions is integrable imply

$$(uv)' = u'v + uv'$$

is integrable on $[a, b]$.

We have

$$\int_a^b [u(x)v'(x) + u'(x)v(x)]\,dx = u(x)v(x)\Big|_a^b,$$

Theorem: Suppose that f' is nonnegative and integrable and g is continuous on $[a, b]$. Then

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx \tag{5.41}$$

for some c in $[a, b]$.

Proof: Since f is differentiable on $[a, b]$, it is continuous on $[a, b]$. Since g is continuous on $[a, b]$, so is fg.

Since we know if f is continuous then f is integrable, thus the integrals in (5.41) exist. If

$$G(x) = \int_a^x g(t)\,dt, \tag{5.42}$$

then $G'(x) = g(x),\ a < x < b$.

Therefore, by using integration by parts with $u = f$ and $v = G$ yields

$$\int_a^b f(x)g(x)\,dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx. \tag{5.43}$$

Since f' is nonnegative and G is continuous, the first mean value theorem of integral implies that

$$\int_a^b f'(x)G(x)\,dx = G(c)\int_a^b f'(x)\,dx \tag{5.44}$$

for some c in $[a,b]$.

But we know that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

From this and (5.42), (5.44) can be rewritten as

$$\int_a^b f'(x)G(x)\,dx = (f(b)-f(a))\int_a^c g(x)\,dx.$$

Substituting this into (5.43) and noting that $G(a) = 0$ yields

$$\begin{aligned}
\int_a^b f(x)g(x)\,dx &= f(b)\int_a^b g(x)\,dx - (f(b)-f(a))\int_a^c g(x)\,dx,\\
&= f(a)\int_a^c g(x)\,dx + f(b)\Bigg(\int_a^b g(x)\,dx \\
&\quad - \int_c^a g(x)\,dx\Bigg)\\
&= f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx.
\end{aligned}$$

5.6 Integration by Substitution

Theorem: Suppose that the transformation $x = \phi(t)$ maps the interval $c \le t \le d$ into the interval $a \le x \le b$, with $\phi(c) = \alpha$ and $\phi(d) = \beta$, and let f be continuous on $[a,b]$.

Let ϕ' be integrable on $[c,d]$. Then

$$\int_\alpha^\beta f(x)\,dx = \int_c^d f(\phi(t))\phi'(t)\,dt. \tag{5.45}$$

Proof: Both integrals in (5.45) exist: the one on the left by the fact that if f is continuous on $[a,b]$, then f is integrable on $[a,b]$, the one on the right by the continuity of $f(\phi(t))$.

The function

$$F(x) = \int_a^x f(y)\,dy$$

is an antiderivative of f on $[a,b]$ and, therefore, also on the closed interval with endpoints α and β.

Hence, by fundamental theorem of calculus,

$$\int_{\alpha}^{\beta} f(x)\,dx = F(\beta) - F(\alpha). \tag{5.46}$$

By the chain rule, the function

$$G(t) = F(\phi(t))$$

is an antiderivative of $f(\phi(t))\phi'(t)$ on $[c, d]$. Therefore, we have

$$\begin{aligned}\int_c^d f(\phi(t))\phi'(t)\,dt &= G(d) - G(c) = F(\phi(d)) - F(\phi(c)) \\ &= F(\beta) - F(\alpha).\end{aligned}$$

Comparing this with (5.46) yields (5.45).

Example: Evaluate the integral

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)(1-x^2)^{-1/2}dx.$$

Solution: We let

$$f(x) = (1-2x^2)(1-x^2)^{-1/2}, \quad -1/\sqrt{2} \le x \le 1/\sqrt{2},$$

and

$$x = \phi(t) = \sin t, \quad -\pi/4 \le t \le \pi/4.$$

Then $\phi'(t) = \cos t$ and

$$\begin{aligned}I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} f(x)\,dx &= \int_{-\pi/4}^{\pi/4} f(\sin t)\cos t\,dt \\ &= \int_{-\pi/4}^{\pi/4} (1-2\sin^2 t)(1-\sin^2 t)^{-1/2}\cos t\,dt.\end{aligned} \tag{5.47}$$

$$\begin{aligned}(1-\sin^2 t)^{-1/2} &= (\cos t)^{-1}, -\pi/4 \le t \le \pi/4 \\ &\text{and} \\ 1 - 2\sin^2 t &= \cos 2t,\end{aligned}$$

(5.47) yields

$$I = \int_{-\pi/4}^{\pi/4} \cos 2t\,dt = \left.\frac{\sin 2t}{2}\right|_{-\pi/4}^{\pi/4} = 1.$$

Example: Evaluate the integral

$$I = \int_0^{5\pi} \frac{\sin t}{2+\cos t}\,dt.$$